第６回 関数空間セミナー 報告集

Seminar on Function Spaces, 1997

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第６回 関数空間セミナー 報告集
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１９９７年１２月２４日（水）～１２月２６日（金）
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目次

Equivalence relations among order preserving operator inequalities and a related open problem

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Abstract

$A \geq B \geq 0$ ensures $A^{1-t} \geq (A^{\frac{-t}{2}} B^p A^{\frac{-t}{2}})^{\frac{1-t}{p-t}}$ for any $p \geq 1$ and $t \leq 0$. This inequality is well known as Furuta inequality. In case $t \geq 0$ and $A \geq B \geq 0$ with $A > 0$, four inequalities similar to this inequality have been obtained by many researchers. In this paper, we show relations among these four inequalities and also we discuss an open problem related to one of four inequalities.

1 Introduction

In what follows, a capital letter means a bounded linear operator on a complex Hilbert space H. An operator T is said to be positive (denoted by: $T \geq 0$) if $(Tx, x) \geq 0$ for all $x \in H$. Also, an operator T is strictly positive (denoted by: $T > 0$) if T is positive and invertible. In 1934, Löwner established the following famous theorem.

Theorem L-H (Löwner-Heinz 1934). *If $A \geq B \geq 0$, then $A^\alpha \geq B^\alpha$ holds for $\alpha \in [0,1]$.*

This theorem is very simple and useful. But the condition $\alpha \in [0,1]$ is too restrictive to apply. As an extension of Theorem L-H, Furuta established the following theorem.

Theorem F (Furuta 1987)[5].

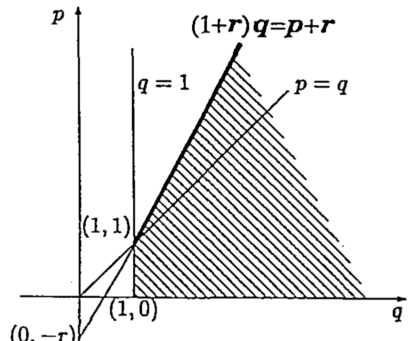
If $A \geq B \geq 0$, then for each $r \geq 0$,

(i) $(B^{\frac{r}{2}} A^p B^{\frac{r}{2}})^{\frac{1}{q}} \geq (B^{\frac{r}{2}} B^p B^{\frac{r}{2}})^{\frac{1}{q}}$

and

(ii) $(A^{\frac{r}{2}} A^p A^{\frac{r}{2}})^{\frac{1}{q}} \geq (A^{\frac{r}{2}} B^p A^{\frac{r}{2}})^{\frac{1}{q}}$

hold for $p \geq 0$ and $q \geq 1$ with $(1+r)q \geq p+r$.



Figure

Theorem F was shown in [5]. Alternative proofs are given in [1] and [8], and one page elementary proof is shown in [6]. It turns out that (i) and (ii) of Theorem F are mutually equivalent. It is shown in [11] that the domain drawn for p, q and r in the figure is best possible one for Theorem F. Put $r = 0$ in Theorem F, then we obtain Theorem L-H, so that Theorem F is an extension of Theorem L-H.

In case $r \leq 0$, we obtain the following Theorem A by many researchers.

Theorem A [9][10]. *Let A, $B \in B(H)$, then the following assertions hold.*

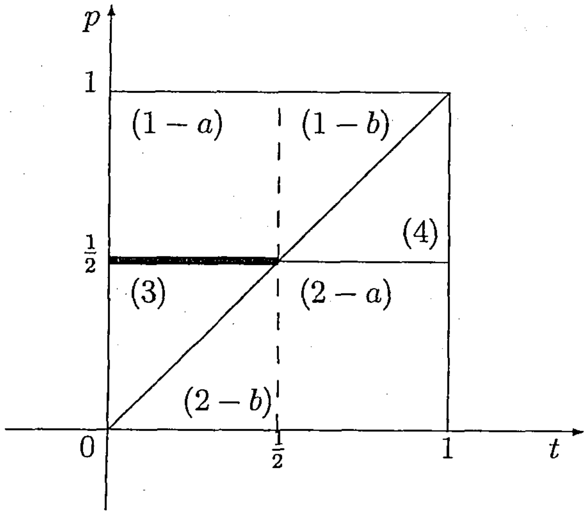
(1) *If $A \geq B \geq 0$ with $A > 0$, then* $A^{1-t} \geq (A^{\frac{-t}{2}} B^p A^{\frac{-t}{2}})^{\frac{1-t}{p-t}}$ *for $1 \geq p > t \geq 0$ with $p \geq \frac{1}{2}$.*

(2) *If $A \geq B \geq 0$ with $A > 0$, then* $A^{-t} \geq (A^{\frac{-t}{2}} B^p A^{\frac{-t}{2}})^{\frac{-t}{p-t}}$ *for $1 \geq t > p \geq 0$ with $\frac{1}{2} \geq p$.*

(3) *If $A \geq B \geq 0$ with $A > 0$, then* $A^{2p-t} \geq (A^{\frac{-t}{2}} B^p A^{\frac{-t}{2}})^{\frac{2p-t}{p-t}}$ *for $\frac{1}{2} \geq p > t \geq 0$.*

(4) *If $A \geq B \geq 0$ with $A > 0$, then* $A^{2p-1-t} \geq (A^{\frac{-t}{2}} B^p A^{\frac{-t}{2}})^{\frac{2p-1-t}{p-t}}$ *for $1 \geq t > p \geq \frac{1}{2}$.*

A part of (1) is shown in [13]. And complete proof of (1) is shown in [2]. Nice proofs of (1) and (3) are shown in [9]. And extensions of (1) and (3) are shown in [3] and [4]. It is shown in [10] that all inequalities hold and also their outside exponents are optimal except (3) called "Mysterious Δ-zone". In this paper, we show some relations among these inequalities in Theorem A. And we show a counterexample related to (3).



Figure

2 Results

At first we can unify four inequalities of Theorem A as follows:

Theorem 1. *If $A \geq B \geq 0$ with $A > 0$, then*

$$A^{q-t} \geq (A^{\frac{-t}{2}} B^p A^{\frac{-t}{2}})^{\frac{q-t}{p-t}} \tag{2.1}$$

holds under the condition (i) *or* (ii);
(i) $2p \geq q \geq p > t \geq 0$ *and* $1 \geq q > 0$,
(ii) $1 \geq t > p \geq q \geq 2p - 1$ *and* $1 > q \geq 0$.

We remark that if we put $q = 1$ and $q = 2p$ in (i) of Theorem 1, then we obtain (1) and (3) in Theorem A respectively. And also if we put $q = 0$ and $q = 2p - 1$ in (ii) of Theorem 1, then we obtain (2) and (4) in Theorem A respectively. The inequality (2.1) is the same form as the following inequality.

Theorem F'. *If $A \geq B \geq 0$, then for each $q \in [0, 1]$,*

$$A^{q-t} \geq (A^{\frac{-t}{2}} B^p A^{\frac{-t}{2}})^{\frac{q-t}{p-t}} \quad \textit{holds for } p \geq q \textit{ and } t \leq 0.$$

Theorem F' is equivalent to Theorem F.

Theorem 2. *If $\frac{1}{2} \geq p > t \geq 0$ and $\alpha > 0$, then there exist $A, B \in B(H)$ such that $A \geq B \geq 0$ with $A > 0$ and*

$$B^{2p+\alpha} \not\geq A^{\frac{t}{2}} (A^{\frac{-t}{2}} B^p A^{\frac{-t}{2}})^{\frac{2p-t+\alpha}{p-t}} A^{\frac{t}{2}}.$$

3 Lemmas

In this section, we cite some lemmas which are needed to prove our results in the previous section.

Lemma F (Furuta 1995)[7]. *Let A be a positive invertible operator, and let B be an invertible operator. For any real number λ,*

$$(BAB^*)^{\lambda} = BA^{\frac{1}{2}} (A^{\frac{1}{2}} B^* B A^{\frac{1}{2}})^{\lambda - 1} A^{\frac{1}{2}} B^*.$$

Lemma 3. *If $A \geq B \geq 0$ with $A > 0$, then*

$$A^{-ps} \geq (A^{\frac{-1}{2}} B^{1-p} A^{\frac{-1}{2}})^s \quad \textit{holds for } s \in [1,2] \textit{ and } \tfrac{1}{s} \geq p \geq 0.$$

Proof. We may assume that A and B are both invertible in the proof. Applying Lemma F and Theorem L-H for $s \in [1,2]$ and $ps \in [0,1]$, we have

$$\begin{aligned}
(A^{\frac{-1}{2}} B^{1-p} A^{\frac{-1}{2}})^s &= A^{\frac{-1}{2}} B^{\frac{1-p}{2}} (B^{\frac{1-p}{2}} A^{-1} B^{\frac{1-p}{2}})^{s-1} B^{\frac{1-p}{2}} A^{\frac{-1}{2}} \quad \text{by Lemma F} \\
&\leq A^{\frac{-1}{2}} B^{\frac{1-p}{2}} (B^{\frac{1-p}{2}} B^{-1} B^{\frac{1-p}{2}})^{s-1} B^{\frac{1-p}{2}} A^{\frac{-1}{2}} \quad \text{by Theorem L-H} \\
&= A^{\frac{-1}{2}} B^{1-ps} A^{\frac{-1}{2}} \\
&\leq A^{\frac{-1}{2}} A^{1-ps} A^{\frac{-1}{2}} \quad \text{by Theorem L-H} \\
&= A^{-ps}.
\end{aligned}$$

Applying Lemma 3 , we obtain the following corollaries.

Corollary 4. *If $A \geq B \geq 0$ with $A > 0$, then*

$$A^{-1} \geq (A^{\frac{-1}{2}} B^{1-p} A^{\frac{-1}{2}})^{\frac{1}{p}} \quad \textit{holds for } p \in [\tfrac{1}{2}, 1].$$

Proof. Put $s = \frac{1}{p}$ in Lemma 3 .

Corollary 5. *If $A \geq B \geq 0$ with $A > 0$, then*

$$A^{-2p} \geq (A^{\frac{-1}{2}} B^{1-p} A^{\frac{-1}{2}})^2 \quad \textit{holds for } p \in [0, \tfrac{1}{2}].$$

Proof. Put $s = 2$ in Lemma 3 .

4 Proofs of results

First of all, we show the relations among four inequalities in Theorem A.
We may assume that A and B are both invertible in the proof.

(A) Proofs of the relations among four triangular zones.
(a) Proof of $(1) \cap (3) \to (4)$.
Applying Corollary 5 to $A \geq B > 0$, we have

$$A^{-2(1-p)} \geq (A^{\frac{-1}{2}} B^p A^{\frac{-1}{2}})^2 \quad \text{for } p \in [\frac{1}{2}, 1]. \tag{4.1}$$

Applying $(1) \cap (3)$ in Theorem A (i.e., put $p = \frac{1}{2}$ in (3).) to (4.1), we have

$$(A^{-2(1-p)})^{1-t_1} \geq \{(A^{-2(1-p)})^{\frac{-t_1}{2}} (A^{\frac{-1}{2}} B^p A^{\frac{-1}{2}})^{\frac{2}{2}} (A^{-2(1-p)})^{\frac{-t_1}{2}}\}^{\frac{1-t_1}{\frac{1}{2}-t_1}} \tag{4.2}$$

for $\frac{1}{2} > t_1 \geq 0$.
Then we have

$$A^{-2(1-p)(1-t_1)} \geq (A^{\frac{2(1-p)t_1-1}{2}} B^p A^{\frac{2(1-p)t_1-1}{2}})^{\frac{1-t_1}{\frac{1}{2}-t_1}}. \tag{4.3}$$

Put $t_1 = \frac{1-t}{2(1-p)}$ in (4.3), we have

$$A^{2p-1-t} \geq (A^{\frac{-t}{2}} B^p A^{\frac{-t}{2}})^{\frac{2p-1-t}{p-t}} \qquad \text{for} \quad 1 \geq t > p \geq \frac{1}{2}$$

by conditions t_1 and p. So the proof of $(1) \cap (3) \rightarrow (4)$ is complete.

(b). Proof of $(4) \rightarrow (1-b)$. Raise each sides of (4) in Theorem A to the power $\frac{t-1}{2p-1-t} \in [0,1]$ by Theorem L-H and taking inverses of both sides, we have

$$A^{1-t} \leq (A^{\frac{-t}{2}} B^p A^{\frac{-t}{2}})^{\frac{1-t}{p-t}}. \tag{4.4}$$

(4.4) is equivalent to the following (4.5) by Lemma F

$$B^{\frac{-p}{2}} A B^{\frac{-p}{2}} \leq (B^{\frac{-p}{2}} A^t B^{\frac{-p}{2}})^{\frac{1-p}{t-p}} \quad \text{for} \quad 1 \geq t > p \geq \frac{1}{2}. \tag{4.5}$$

Therefore we obtain

$$\begin{aligned} B^{1-p} &\leq B^{\frac{-p}{2}} A B^{\frac{-p}{2}} \quad \text{by } A \geq B \\ &\leq (B^{\frac{-p}{2}} A^t B^{\frac{-p}{2}})^{\frac{1-p}{t-p}}. \end{aligned} \tag{4.6}$$

Put $p_1 = t$ and $t_1 = p$ in (4.6), we have

$$B^{1-t_1} \leq (B^{\frac{-t_1}{2}} A^{p_1} B^{\frac{-t_1}{2}})^{\frac{1-t_1}{p_1-t_1}} \quad \text{for} \quad 1 \geq p_1 > t_1 \geq \frac{1}{2}. \tag{4.7}$$

(4.7) is equivalent to $(1-b)$. So the proof of $(4) \rightarrow (1-b)$ is complete.

(c). Proof of $(1-b) \rightarrow (2-b)$. Applying Corollary 4 to $A \geq B > 0$, we have

$$A^{-1} \geq (A^{\frac{-1}{2}} B^{1-p} A^{\frac{-1}{2}})^{\frac{1}{p}} \quad \text{for} \quad p \in [\frac{1}{2}, 1]. \tag{4.8}$$

Applying $(1-b)$ in Theorem A to the (4.8), we have

$$(A^{-1})^{1-t_1} \geq \{(A^{-1})^{\frac{-t_1}{2}} (A^{\frac{-1}{2}} B^{1-p} A^{\frac{-1}{2}})^{\frac{p_1}{p}} (A^{-1})^{\frac{-t_1}{2}}\}^{\frac{1-t_1}{p_1-t_1}} \tag{4.9}$$

for $1 \geq p_1 > t_1 \geq \frac{1}{2}$.

Put $p_1 = p$ in (4.9). Then we have

$$A^{-(1-t_1)} \geq (A^{\frac{-(1-t_1)}{2}} B^{1-p} A^{\frac{-(1-t_1)}{2}})^{\frac{1-t_1}{p-t_1}}. \tag{4.10}$$

Put $t_2 = 1 - t_1$ and $p_2 = 1 - p$ in (4.10). Then we have

$$A^{-t_2} \geq (A^{\frac{-t_2}{2}} B^{p_2} A^{\frac{-t_2}{2}})^{\frac{-t_2}{p_2-t_2}} \quad \text{for} \quad \frac{1}{2} \geq t_2 > p_2 \geq 0.$$

So the proof of $(1-b) \rightarrow (2-b)$ is complete.

(d). Proof of $(2-b) \rightarrow (3)$. $(2-b)$ is equivalent to the following form:

$$(2-b): A \geq B > 0 \Rightarrow B^{-p} \leq (B^{\frac{-p}{2}} A^t B^{\frac{-p}{2}})^{\frac{-p}{t-p}} \quad \text{for} \quad \frac{1}{2} \geq p > t \geq 0. \tag{4.11}$$

Then applying Lemma F and Theorem L-H, we obtain

$$\begin{aligned} (A^{\frac{-t}{2}} B^p A^{\frac{-t}{2}})^{\frac{2p-t}{p-t}} &= A^{\frac{-t}{2}} B^{\frac{p}{2}} (B^{\frac{p}{2}} A^{-t} B^{\frac{p}{2}})^{\frac{p}{p-t}} B^{\frac{p}{2}} A^{\frac{-t}{2}} \quad \text{by Lemma F} \\ &= A^{\frac{-t}{2}} B^{\frac{p}{2}} \{(B^{\frac{-p}{2}} A^t B^{\frac{-p}{2}})^{\frac{-p}{t-p}}\}^{-1} B^{\frac{p}{2}} A^{\frac{-t}{2}} \\ &\leq A^{\frac{-t}{2}} B^{\frac{p}{2}} B^p B^{\frac{p}{2}} A^{\frac{-t}{2}} \quad \text{by (4.11)} \\ &= A^{\frac{-t}{2}} B^{2p} A^{\frac{-t}{2}} \\ &\leq A^{2p-t} \quad \text{for} \quad \frac{1}{2} \geq p > t \geq 0. \quad \text{by Theorem L-H} \end{aligned}$$

So the proof of $(2-b) \rightarrow (3)$ is complete. Consequently the proofs of the relations among four triangular zones are complete.

(B) Secondly, we show $(1-a) \leftrightarrow (2-a)$.

(e). Proof of $(1-a) \rightarrow (2-a)$. Raise each sides of $(1-a)$ to the power $\frac{t}{1-t} \in [0,1]$ by Theorem L-H, we have

$$A^t \geq (A^{\frac{-t}{2}} B^p A^{\frac{-t}{2}})^{\frac{t}{p-t}}. \tag{4.12}$$

(4.12) is equivalent to the following (4.13) by Lemma F and taking inverses of both sides:

$$B^{-p} \leq (B^{\frac{-p}{2}} A^t B^{\frac{-p}{2}})^{\frac{-p}{t-p}}. \tag{4.13}$$

Put $p_1 = t$ and $t_1 = p$ in (4.13), we have

$$B^{-t_1} \leq (B^{\frac{-t_1}{2}} A^{p_1} B^{\frac{-t_1}{2}})^{\frac{-t_1}{p_1-t_1}} \tag{4.14}$$

for $1 \geq t_1 \geq \frac{1}{2} \geq p_1 \geq 0$ with $t_1 \neq p_1$.

(4.14) is equivalent to the $(2-a)$. So the proof of $(1-a) \rightarrow (2-a)$ is complete.

(f). Proof of $(2-a) \rightarrow (1-a)$. Raise each sides of $(2-a)$ to the power $\frac{1-t}{t} \in [0,1]$ by Theorem L-H,

$$A^{-(1-t)} \geq (A^{\frac{-t}{2}} B^p A^{\frac{-t}{2}})^{\frac{-(1-t)}{p-t}}. \tag{4.15}$$

(4.15) is equivalent to the following (4.16) by Lemma F and taking inverses of both sides:

$$B^{\frac{-p}{2}} A B^{\frac{-p}{2}} \leq (B^{\frac{-p}{2}} A^t B^{\frac{-p}{2}})^{\frac{1-p}{t-p}}. \tag{4.16}$$

Applying the hypothesis $A \geq B$ to (4.16), we have

$$\begin{aligned} B^{1-p} &\leq B^{\frac{-p}{2}} A B^{\frac{-p}{2}} \quad \text{by } A \geq B \\ &\leq (B^{\frac{-p}{2}} A^t B^{\frac{-p}{2}})^{\frac{1-p}{t-p}}. \end{aligned} \tag{4.17}$$

Put $p_1 = t$ and $t_1 = p$ in (4.17), we have

$$B^{1-t_1} \leq (B^{\frac{-t_1}{2}} A^{p_1} B^{\frac{-t_1}{2}})^{\frac{1-t_1}{p_1-t_1}} \tag{4.18}$$

for $1 \geq p_1 \geq \frac{1}{2} \geq t_1 \geq 0$ with $p_1 \neq t_1$.
(4.18) is equivalent to $(1-a)$. So the proof of $(2-a) \rightarrow (1-a)$ is complete. Consequently the proof of $(1-a) \leftrightarrow (2-a)$ is complete.

Proof of Theorem 1. Case (i). Applying Theorem L-H to the hypothesis $A \geq B \geq 0$. Then we obtain $A^q \geq B^q$ for $1 \geq q > 0$. Moreover we apply (1) in Theorem A to A^q and B^q, then we have

$$(A^q)^{1-t_1} \geq \{(A^q)^{\frac{-t_1}{2}} (B^q)^{p_1} (A^q)^{\frac{-t_1}{2}}\}^{\frac{1-t_1}{p_1-t_1}} \tag{4.19}$$

for $1 \geq p_1 > t_1 \geq 0$ with $p_1 \geq \frac{1}{2}$.
This condition is equivalent to the following (4.20):

$$2p_1 \geq 1 \geq p_1 > t_1 \geq 0. \tag{4.20}$$

Put $p_1 = \frac{p}{q}$ and $t_1 = \frac{t}{q}$ in (4.19), we have

$$A^{q-t} \geq (A^{\frac{-t}{2}} B^p A^{\frac{-t}{2}})^{\frac{q-t}{p-t}} \tag{4.21}$$

for $2p \geq q \geq p > t \geq 0$ and $1 \geq q > 0$.

So the proof of (i) is complete.

Case (ii). Applying Lemma 3 to the hypothesis $A \geq B \geq 0$ with $A > 0$, we have

$$A^{-ps} \geq (A^{\frac{-1}{2}} B^{1-p} A^{\frac{-1}{2}})^{s} \tag{4.22}$$

for any $s \in [1, 2]$ and $\frac{1}{s} \geq p \geq 0$.

Put $s = \frac{q}{p}$ in (4.22), we have

$$A^{-q} \geq (A^{\frac{-1}{2}} B^{1-p} A^{\frac{-1}{2}})^{\frac{q}{p}} \tag{4.23}$$

for any $\frac{q}{p} \in [1, 2]$ and $1 \geq q > 0$.

Applying (1) in Theorem A to (4.23), then we have

$$(A^{-q})^{(1-t_1)} \geq \{(A^{-q})^{\frac{-t_1}{2}} (A^{\frac{-1}{2}} B^{1-p} A^{\frac{-1}{2}})^{\frac{q}{p} p_1} (A^{-q})^{\frac{-t_1}{2}}\}^{\frac{1-t_1}{p_1 - t_1}} \tag{4.24}$$

for $2p_1 \geq 1 \geq p_1 > t_1 \geq 0$.

Put $p_1 = \frac{p}{q}$ and $t_1 = \frac{t}{q}$ in (4.24), we have

$$A^{-q+t} \geq (A^{\frac{-(1-t)}{2}} B^{1-p} A^{\frac{-(1-t)}{2}})^{\frac{q-t}{p-t}} \tag{4.25}$$

for $2p \geq q \geq p > t \geq 0$ and $1 \geq q > 0$.

Put $q_2 = 1 - q$, $p_2 = 1 - p$ and $t_2 = 1 - t$ in (4.25), we have

$$A^{q_2 - t_2} \geq (A^{\frac{-t_2}{2}} B^{p_2} A^{\frac{-t_2}{2}})^{\frac{q_2 - t_2}{p_2 - t_2}} \tag{4.26}$$

for $1 \geq t_2 > p_2 \geq q_2 \geq 2p_2 - 1$ and $1 > q_2 \geq 0$.

So the proof of (ii) is complete. Hence the proof of Theorem 1 is complete.

Remark. (1) in Theorem A is essential to the proof of Theorem 1. We consider (1) in Theorem A as follows.

Theorem A-s (satellite version).

(1) *If $A \geq B \geq 0$ with $A > 0$, then* $A \geq B \geq A^{\frac{t}{2}} (A^{\frac{-t}{2}} B^p A^{\frac{-t}{2}})^{\frac{1-t}{p-t}} A^{\frac{t}{2}}$ *for* $1 \geq p > t \geq 0$ *with* $p \geq \frac{1}{2}$.

In the same way as the proof of Theorem 1, we obtain the following theorem.

Theorem 1-s.

(i) *If $A \geq B \geq 0$ with $A > 0$, then* $A^q \geq B^q \geq A^{\frac{t}{2}} (A^{\frac{-t}{2}} B^p A^{\frac{-t}{2}})^{\frac{q-t}{p-t}} A^{\frac{t}{2}}$

holds for any $2p \geq q \geq p > t \geq 0$ *and* $1 \geq q > 0$.

(ii) *If $A \geq B > 0$ with $A > 0$, then*

$$\begin{aligned} A^q &\geq A^{\frac{1}{2}} (A^{\frac{-1}{2}} B^p A^{\frac{-1}{2}})^{\frac{q-1}{p-1}} A^{\frac{1}{2}} \\ &\geq A^{\frac{t}{2}} (A^{\frac{-t}{2}} B^p A^{\frac{-t}{2}})^{\frac{q-t}{p-t}} A^{\frac{t}{2}} \end{aligned}$$

holds for any $1 \geq t > p \geq q \geq 2p - 1$ *and* $1 > q \geq 0$.

Proof of Theorem 2. We may assume that A and B are both invertible in the proof and the following inequality holds for any $\alpha = \alpha(p, t) > 0$:

$$B^{2p+\alpha} \geq A^{\frac{t}{2}} (A^{\frac{-t}{2}} B^p A^{\frac{-t}{2}})^{\frac{2p-t+\alpha}{p-t}} A^{\frac{t}{2}} \tag{4.27}$$

for fixed p and t such that $\frac{1}{2} \geq p > t \geq 0$.

(4.27) is equivalent to the following (4.28) by Lemma F and taking inverses of both sides:

$$B^{-(p+\alpha)} \leq (B^{\frac{-p}{2}} A^t B^{\frac{-p}{2}})^{\frac{-(p+\alpha)}{t-p}}. \tag{4.28}$$

Put $p_1 = t$ and $t_1 = p$ in (4.28), we have

$$B^{-(t_1+\alpha)} \leq (B^{\frac{-t_1}{2}} A^{p_1} B^{\frac{-t_1}{2}})^{\frac{-(t_1+\alpha)}{p_1-t_1}} \tag{4.29}$$

for fixed t_1 and p_1 such that $\frac{1}{2} \geq t_1 > p_1 \geq 0$.
But this is a contradiction by the best possibility of $(2-b)$ in Theorem A [10]. So that the proof of Theorem 2 is complete.

We obtain the following assertion by Theorem 2.

Corollary 6. *If $\frac{1}{2} \geq p > t \geq 0$, then there exist $A, B \in B(H)$ such that $A \geq B \geq 0$ with $A > 0$ and*

$$B \not\geq A^{\frac{t}{2}}(A^{\frac{-t}{2}} B^p A^{\frac{-t}{2}})^{\frac{1-t}{p-t}} A^{\frac{t}{2}}.$$

Its concrete example is given by J-F-Jiang.

5 Examples

In this section, we show an example related to the best possibility of (3) in Theorem A. Does the following conjecture hold?

Conjecture. *If $A \geq B \geq 0$ with $A > 0$, then*

$$A \geq A^{\frac{t}{2}}(A^{\frac{-t}{2}} B^p A^{\frac{-t}{2}})^{\frac{1-t}{p-t}} A^{\frac{t}{2}} \qquad \text{for any } \frac{1}{2} \geq p > t \geq 0. \tag{5.1}$$

T.Furuta expected that this conjecture did not hold, and (3) in Theorem A was the best possible. We obtain the following example.

Theorem 7 (Counterexample). *If $p = 0.3$ and $t = 0.15$, then there exist $A, B \in B(H)$ such that $A \geq B \geq 0$ with $A > 0$ and*

$$A \not\geq A^{\frac{t}{2}}(A^{\frac{-t}{2}} B^p A^{\frac{-t}{2}})^{\frac{1-t}{p-t}} A^{\frac{t}{2}}$$

Example. We defined by X and Y as follows for any $1 - 2p \geq \alpha > 0$.

$$X \equiv A^{2p+\alpha} - A^{\frac{t}{2}}(A^{\frac{-t}{2}} B^p A^{\frac{-t}{2}})^{\frac{2p-t+\alpha}{p-t}} A^{\frac{t}{2}}.$$

$$Y \equiv B^{2p+\alpha} - A^{\frac{t}{2}}(A^{\frac{-t}{2}} B^p A^{\frac{-t}{2}})^{\frac{2p-t+\alpha}{p-t}} A^{\frac{t}{2}}.$$

And、 A, B, p and t are defined by

$$A \equiv \begin{pmatrix} 18926 & 2549 & 26988 \\ 2549 & 38479 & 3638 \\ 26988 & 3638 & 38524 \end{pmatrix} \geq \begin{pmatrix} 19 & 0 & 0 \\ 0 & 38133 & 0 \\ 0 & 0 & 1 \end{pmatrix} \equiv B.$$

$p = 0.3$ and $t = 0.15$.

(1). In case $\alpha = 1 - 2p = 0.4$.

$$X = \begin{pmatrix} 18916.25\cdots & 2587.35\cdots & 26990.14\cdots \\ 2587.35\cdots & 432.25\cdots & 3655.53\cdots \\ 26990.14\cdots & 3655.53\cdots & 38523.50\cdots \end{pmatrix}.$$

Eigenvalues of X are $57785.0756\cdots$, $87.9132\cdots$ and $-0.9723\cdots$. So that $X \not\geq 0$.

$$Y = \begin{pmatrix} 9.2543\cdots & 38.3541\cdots & 2.1437\cdots \\ 38.3541\cdots & 86.2527\cdots & 17.5346\cdots \\ 2.1437\cdots & 17.5346\cdots & 0.5094\cdots \end{pmatrix}.$$

Eigenvalues of Y are $104.8795\cdots$, $-9.5621\cdots$ and $0.6990\cdots$. So that $Y \not\geq 0$.

(2). In case $0.37 = \alpha < 1 - 2p = 0.4$.

$$X = \begin{pmatrix} 13614.65\cdots & 1817.80\cdots & 19425.33\cdots \\ 1817.80\cdots & 300.01\cdots & 2567.35\cdots \\ 19425.33\cdots & 2567.35\cdots & 27728.05\cdots \end{pmatrix}.$$

Eigenvalues of X are $41578.4615\cdots$, $64.2655\cdots$ and $-0.0014\cdots$. So that $X \not\geq 0$.

$$Y = \begin{pmatrix} 8.2638\cdots & 27.9370\cdots & 2.0135\cdots \\ 27.9370\cdots & 60.1869\cdots & 12.7824\cdots \\ 2.0135\cdots & 12.7824\cdots & 0.5413\cdots \end{pmatrix}.$$

Eigenvalues of Y are $74.4826\cdots$, $-6.1697\cdots$ and $0.6733\cdots$. So that $Y \not\geq 0$.

Therefore we expect the following conjecture. But at present we do not obtain its proof.

Conjecture. *If $\frac{1}{2} \geq p > t \geq 0$ and $\alpha > 0$, then there exist $A, B \in B(H)$ such that*

$$A^{2p+\alpha} \not\geq A^{\frac{t}{2}}(A^{\frac{-t}{2}}B^pA^{\frac{-t}{2}})^{\frac{2p-t+\alpha}{p-t}}A^{\frac{t}{2}}.$$

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Generalized operator functions implying order preserving operator inequalities

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1 Introduction

This report is based on the following preprint:

T.Furuta, T.Yamazaki and M.Yanagida, *Operator functions implying generalized Furuta inequality*, to appear in Mathematical Inequalities and Applications **1** (1998).

A capital letter means a bounded linear operator on a complex Hilbert space H. An operator T is said to be positive (denoted by $T \geq 0$) if $(Tx, x) \geq 0$ for all $x \in H$ and also an operator T is said to be strictly positive (denoted by $T > 0$) if T is positive and invertible. The following Theorem F is an extension of the celebrated Löwner-Heinz theorem: $A \geq B \geq 0$ *ensures* $A^{\alpha} \geq B^{\alpha}$ *for any* $\alpha \in [0, 1]$.

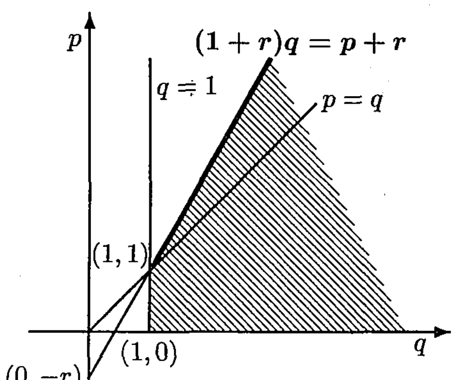
Theorem F ([6]).
If $A \geq B \geq 0$, *then for each* $r \geq 0$,

(i) $\qquad (B^{\frac{r}{2}} A^p B^{\frac{r}{2}})^{\frac{1}{q}} \geq (B^{\frac{r}{2}} B^p B^{\frac{r}{2}})^{\frac{1}{q}}$

and

(ii) $\qquad (A^{\frac{r}{2}} A^p A^{\frac{r}{2}})^{\frac{1}{q}} \geq (A^{\frac{r}{2}} B^p A^{\frac{r}{2}})^{\frac{1}{q}}$

hold for $p \geq 0$ *and* $q \geq 1$ *with* $(1+r)q \geq p+r$.



Figure

We remark that Theorem F yields Löwner-Heinz theorem when we put $r = 0$ in (i) or (ii) stated above. Alternative proofs of Theorem F are given in [3][13] and also an elementary one-page proof in [7]. It is shown in [14] that the domain drawn for p, q and r in the Figure is best possible one for Theorem F. Since now, many applications of Theorem F have been developed by many researchers in the following branches.

APPLICATIONS OF THEOREM F

(A) OPERATOR INEQUALITIES

(1) Characterizations of operators satisfying $\log A \geq \log B$

(2) Generalizations of Ando's theorem

(3) Other order preserving operator inequalities

(4) Applications to the relative operator entropy

(5) Applications to Ando-Hiai log majorization

(6) Generalized Aluthge transformation

(B) NORM INEQUALITIES

(1) Several generalizations of Heinz-Kato theorem

(2) Generalizations of some theorems on norms

(3) An extension of Kosaki trace inequality and parallel results

(C) OPERATOR EQUATIONS

(1) Generalizations of Pedersen-Takesaki theorem and related results

In [10] we established the following Theorem G as extensions of Theorem F.

Theorem G ([10]). *If $A \geq B \geq 0$ with $A > 0$, then for each $t \in [0,1]$ and $p \geq 1$,*

$$F_{p,t}(A,B,r,s) = A^{\frac{-r}{2}}\{A^{\frac{r}{2}}(A^{\frac{-t}{2}}B^pA^{\frac{-t}{2}})^sA^{\frac{r}{2}}\}^{\frac{1-t+r}{(p-t)s+r}}A^{\frac{-r}{2}}$$

is decreasing for $r \geq t$ and $s \geq 1$, and $F_{p,t}(A,A,r,s) \geq F_{p,t}(A,B,r,s)$, that is, for each $t \in [0,1]$ and $p \geq 1$,

$$A^{1-t+r} \geq \{A^{\frac{r}{2}}(A^{\frac{-t}{2}}B^pA^{\frac{-t}{2}})^sA^{\frac{r}{2}}\}^{\frac{1-t+r}{(p-t)s+r}}$$

holds for any $s \geq 1$ and $r \geq t$.

Ando-Hiai[2] established excellent log majorization results and proved the following useful inequality equivalent to the main log majorization theorem: *If $A \geq B \geq 0$ with $A > 0$, then*

$$A^r \geq \{A^{\frac{r}{2}}(A^{\frac{-1}{2}}B^pA^{\frac{-1}{2}})^rA^{\frac{r}{2}}\}^{\frac{1}{p}}$$

holds for any $p \geq 1$ and $r \geq 1$. Theorem G interpolates the inequality stated above by Ando-Hiai and Theorem F itself, and also extends results of [4][8] and [9]. Recently a nice mean theoretic proof of Theorem G is shown in [5] and the result on the best possibility of Theorem G is shown in [15].

Very recently the following Theorem H is obtained in [11] as an extension of Theorem G and a simplified proof of Theorem H is shown in [12].

Theorem H ([11]). *Let $A \geq B \geq 0$ with $A > 0$. For each $t \in [0,1], q \geq 0$ and $p \geq \max\{q,t\}$,*

$$G_{p,q,t}(A,B,r,s) = A^{\frac{-r}{2}}\{A^{\frac{r}{2}}(A^{\frac{-t}{2}}B^pA^{\frac{-t}{2}})^sA^{\frac{r}{2}}\}^{\frac{q-t+r}{(p-t)s+r}}A^{\frac{-r}{2}}$$

is decreasing for $r \geq t$ and $s \geq 1$.

Here we show Theorem 1 by using Theorem F and we show Theorem 2, which is an extension of Theorem H, and Corollary 3 by using Theorem 1.

2 Results

Theorem 1. *Let A and B be positive invertible operators satisfying*

$$A \geq (A^{\frac{1}{2}}BA^{\frac{1}{2}})^{\frac{\beta_0}{\alpha_0+\beta_0}} \quad \text{for fixed } \alpha_0 \geq 0 \text{ and } \beta_0 \geq 0 \text{ with } \alpha_0+\beta_0>0.$$

Then the following (i) *and* (ii) *hold and they are mutually equivalent:*

(i) *For any fixed $\delta \geq -\beta_0$,*

$$f(\lambda,\mu)=A^{\frac{-\mu}{2}}(A^{\frac{\mu}{2}}B^{\lambda}A^{\frac{\mu}{2}})^{\frac{\delta+\beta_0\mu}{\alpha_0\lambda+\beta_0\mu}}A^{\frac{-\mu}{2}}$$

is decreasing for $\mu \geq 1$ and $\lambda \geq 1$ such that $\alpha_0\lambda \geq \delta$.

(ii) *For any fixed $\delta \leq \alpha_0$,*

$$f(\lambda,\mu)=A^{\frac{-\mu}{2}}(A^{\frac{\mu}{2}}B^{\lambda}A^{\frac{\mu}{2}})^{\frac{\delta+\beta_0\mu}{\alpha_0\lambda+\beta_0\mu}}A^{\frac{-\mu}{2}}$$

is decreasing for $\lambda \geq 1$ and $\mu \geq 1$ such that $\beta_0\mu \geq -\delta$.

Applying Theorem 1, we obtain the following extension of Theorem H.

Theorem 2. *Let $A \geq B \geq 0$ with $A>0$. For each $t \in [0,1]$ and $p \geq t$, the following* (i) *and* (ii) *hold and they are mutually equivalent:*

(i) *If $q \geq 0$, then*

$$G_{p,q,t}(A,B,r,s)=A^{\frac{-r}{2}}\{A^{\frac{r}{2}}(A^{\frac{-t}{2}}B^{p}A^{\frac{-t}{2}})^{s}A^{\frac{r}{2}}\}^{\frac{q-t+r}{(p-t)s+r}}A^{\frac{-r}{2}}$$

is decreasing for $r \geq t$ and $s \geq 1$ such that $(p-t)s \geq q-t$.

(ii) *If $p \geq q$, then*

$$G_{p,q,t}(A,B,r,s)=A^{\frac{-r}{2}}\{A^{\frac{r}{2}}(A^{\frac{-t}{2}}B^{p}A^{\frac{-t}{2}})^{s}A^{\frac{r}{2}}\}^{\frac{q-t+r}{(p-t)s+r}}A^{\frac{-r}{2}}$$

is decreasing for $s \geq 1$ and $r \geq \max\{t, t-q\}$.

We write $A \gg B$ if $\log A \geq \log B$ for positive invertible operator A and B which is called chaotic order [4] and related results on chaotic order are discussed in [1] and [4]. Also Theorem 1 implies the following characterization of chaotic order.

Corollary 3. *The following assertions are mutually equivalent:*

(i) *$A \gg B$ (i.e., $\log A \geq \log B$).*

(ii) *For any fixed $q \geq 0$,*

$$F_q(p,r)=A^{\frac{-r}{2}}(A^{\frac{r}{2}}B^{p}A^{\frac{r}{2}})^{\frac{q+r}{p+r}}A^{\frac{-r}{2}}$$

is decreasing for $p \geq q$ and $r \geq 0$.

(iii) *For any fixed $q \leq 0$,*

$$F_q(p,r)=A^{\frac{-r}{2}}(A^{\frac{r}{2}}B^{p}A^{\frac{r}{2}})^{\frac{q+r}{p+r}}A^{\frac{-r}{2}}$$

is decreasing for $p \geq 0$ and $r \geq -q$.

The equivalence relation between (i) and (ii) is shown in [4][9].

3 Proofs of results

We need the following lemmas to give proofs of the results.

Lemma F (Furuta lemma[10]). *Let $A > 0$ and B be an invertible operator. Then*

$$(BAB^*)^\lambda = BA^{\frac{1}{2}}(A^{\frac{1}{2}}B^*BA^{\frac{1}{2}})^{\lambda-1}A^{\frac{1}{2}}B^*$$

holds for any real number λ.

Lemma 1. *Let A and B be positive invertible operators satisfying*

$$A \geq (A^{\frac{1}{2}}BA^{\frac{1}{2}})^{\frac{\beta_0}{\alpha_0+\beta_0}} \quad \textit{for fixed } \alpha_0 \geq 0 \textit{ and } \beta_0 \geq 0 \textit{ with } \alpha_0+\beta_0 > 0. \tag{3.1}$$

Then the following inequality holds:

$$A^\mu \geq (A^{\frac{\mu}{2}}B^\lambda A^{\frac{\mu}{2}})^{\frac{\beta_0\mu}{\alpha_0\lambda+\beta_0\mu}} \quad \textit{for } \lambda \geq 1 \textit{ and } \mu \geq 1. \tag{3.2}$$

Proof of Lemma 1. In case $\beta_0 = 0$, (3.1) means $A \geq I$, obviously $A^\mu \geq I$ holds for any $\mu \geq 1$, so that (3.2) holds. In case $\alpha_0 = 0$, (3.1) means $I \geq B$, obviously $I \geq B^\lambda$ holds for any $\lambda \geq 1$, so that (3.2) holds, too. Therefore we have only to consider the case $\alpha > 0$ and $\beta > 0$. Applying (ii) of Theorem F to (3.1), we have

$$A^{1+r_1} \geq \{A^{\frac{r_1}{2}}(A^{\frac{1}{2}}BA^{\frac{1}{2}})^{\frac{\beta_0 p_1}{\alpha_0+\beta_0}}A^{\frac{r_1}{2}}\}^{\frac{1+r_1}{p_1+r_1}} \quad \text{for any } p_1 \geq 1 \text{ and } r_1 \geq 0. \tag{3.3}$$

Putting $p_1 = \frac{\alpha_0+\beta_0}{\beta_0} \geq 1$ in (3.3), we have

$$A^{1+r_1} \geq (A^{\frac{1}{2}(1+r_1)}BA^{\frac{1}{2}(1+r_1)})^{\frac{(1+r_1)\beta_0}{\alpha_0+\beta_0+\beta_0 r_1}} \quad \text{for any } r_1 \geq 0. \tag{3.4}$$

Put $\mu = 1 + r_1 \geq 1$ in (3.4), then we have

$$A^\mu \geq (A^{\frac{\mu}{2}}BA^{\frac{\mu}{2}})^{\frac{\beta_0\mu}{\alpha_0+\beta_0\mu}} \quad \text{for } \mu \geq 1. \tag{3.5}$$

(3.5) is equivalent to the following (3.6) by Lemma F:

$$(B^{\frac{1}{2}}A^\mu B^{\frac{1}{2}})^{\frac{\alpha_0}{\alpha_0+\beta_0\mu}} \geq B \quad \text{for } \mu \geq 1. \tag{3.6}$$

Again applying (i) of Theorem F to (3.6), we have

$$\{B^{\frac{r_2}{2}}(B^{\frac{1}{2}}A^\mu B^{\frac{1}{2}})^{\frac{\alpha_0 p_2}{\alpha_0+\beta_0\mu}}B^{\frac{r_2}{2}}\}^{\frac{1+r_2}{p_2+r_2}} \geq B^{1+r_2} \quad \text{for any } p_2 \geq 1 \text{ and } r_2 \geq 0. \tag{3.7}$$

Putting $p_2 = \frac{\alpha_0+\beta_0\mu}{\alpha_0} \geq 1$ in (3.7), we have

$$(B^{\frac{1}{2}(1+r_2)}A^\mu B^{\frac{1}{2}(1+r_2)})^{\frac{(1+r_2)\alpha_0}{\alpha_0+\beta_0\mu+\alpha_0 r_2}} \geq B^{1+r_2} \quad \text{for any } r_2 \geq 0. \tag{3.8}$$

Put $\lambda = 1 + r_2 \geq 1$ in (3.8), then we have

$$(B^{\frac{\lambda}{2}}A^\mu B^{\frac{\lambda}{2}})^{\frac{\alpha_0\lambda}{\alpha_0\lambda+\beta_0\mu}} \geq B^\lambda \quad \text{for } \lambda \geq 1 \text{ and } \mu \geq 1. \tag{3.9}$$

Hence the proof of Lemma 1 is complete since (3.9) is equivalent to (3.2) by Lemma F. □

Proof of Theorem 1.
Proof of (i). We recall the following condition on $\delta, \alpha_0, \beta_0$ and λ in (i):

$$\text{for any fixed } \delta \geq -\beta_0 \text{ and } \lambda \geq 1 \text{ such that } \alpha_0\lambda \geq \delta. \tag{3.10}$$

(a) *Proof of the result that $f(\lambda,\mu)$ is decreasing for $\lambda \geq 1$ such that $\alpha\lambda \geq \delta$.*
The hypothesis in Theorem 1 ensures (3.2) by Lemma 1:

$$A^{\mu} \geq (A^{\frac{\mu}{2}}B^{\lambda}A^{\frac{\mu}{2}})^{\frac{\beta_0\mu}{\alpha_0\lambda+\beta_0\mu}} \quad \text{for } \lambda \geq 1 \text{ and } \mu \geq 1, \tag{3.2}$$

and (3.2) is equivalent to the following (3.9) by Lemma F:

$$(B^{\frac{\lambda}{2}}A^{\mu}B^{\frac{\lambda}{2}})^{\frac{\alpha_0\lambda}{\alpha_0\lambda+\beta_0\mu}} \geq B^{\lambda} \quad \text{for } \lambda \geq 1 \text{ and } \mu \geq 1. \tag{3.9}$$

(3.9) yields the following (3.11) by Löwner-Heinz theorem:

$$(B^{\frac{\lambda}{2}}A^{\mu}B^{\frac{\lambda}{2}})^{\frac{\alpha_0 w}{\alpha_0\lambda+\beta_0\mu}} \geq B^{w} \quad \text{for } \lambda \geq 1, \mu \geq 1 \text{ and any } w \text{ such that } \lambda \geq w \geq 0. \tag{3.11}$$

Define $g(\lambda) = (A^{\frac{\mu}{2}}B^{\lambda}A^{\frac{\mu}{2}})^{\frac{\delta+\beta_0\mu}{\alpha_0\lambda+\beta_0\mu}}$. Then $f(\lambda,\mu) = A^{\frac{-\mu}{2}}g(\lambda)A^{\frac{-\mu}{2}}$ and we have

$$\begin{aligned}
g(\lambda) =&(A^{\frac{\mu}{2}}B^{\lambda}A^{\frac{\mu}{2}})^{\frac{\delta+\beta_0\mu}{\alpha_0\lambda+\beta_0\mu}} \\
=&\{(A^{\frac{\mu}{2}}B^{\lambda}A^{\frac{\mu}{2}})^{\frac{\alpha_0\lambda+\beta_0\mu+\alpha_0 w}{\alpha_0\lambda+\beta_0\mu}}\}^{\frac{\delta+\beta_0\mu}{\alpha_0\lambda+\beta_0\mu+\alpha_0 w}} \\
=&\{A^{\frac{\mu}{2}}B^{\frac{\lambda}{2}}(B^{\frac{\lambda}{2}}A^{\mu}B^{\frac{\lambda}{2}})^{\frac{\alpha_0 w}{\alpha_0\lambda+\beta_0\mu}}B^{\frac{\lambda}{2}}A^{\frac{\mu}{2}}\}^{\frac{\delta+\beta_0\mu}{\alpha_0\lambda+\beta_0\mu+\alpha_0 w}} \qquad \text{by Lemma F} \\
\geq&(A^{\frac{\mu}{2}}B^{\frac{\lambda}{2}}B^{w}B^{\frac{\lambda}{2}}A^{\frac{\mu}{2}})^{\frac{\delta+\beta_0\mu}{\alpha_0\lambda+\beta_0\mu+\alpha_0 w}} \\
=&(A^{\frac{\mu}{2}}B^{\lambda+w}A^{\frac{\mu}{2}})^{\frac{\delta+\beta_0\mu}{\alpha_0(\lambda+w)+\beta_0\mu}} \\
=&g(\lambda+w).
\end{aligned}$$

The last inequality holds by (3.11) and Löwner-Heinz theorem since $\frac{\delta+\beta_0\mu}{\alpha_0\lambda+\beta_0\mu+\alpha_0 w} \in [0,1]$ holds by the condition (3.10). Hence $f(\lambda,\mu) = A^{\frac{-\mu}{2}}g(\lambda)A^{\frac{-\mu}{2}}$ is decreasing for $\lambda \geq 1$ such that $\alpha\lambda \geq \delta$.
(b) *Proof of the result that $f(\lambda,\mu)$ is decreasing for $\mu \geq 1$.*
By Lemma F,

$$\begin{aligned}
f(\lambda,\mu) &= A^{\frac{-\mu}{2}}(A^{\frac{\mu}{2}}B^{\lambda}A^{\frac{-\mu}{2}})^{\frac{\delta+\beta_0\mu}{\alpha_0\lambda+\beta_0\mu}}A^{\frac{-\mu}{2}} \\
&= B^{\frac{\lambda}{2}}(B^{\frac{\lambda}{2}}A^{\mu}B^{\frac{\lambda}{2}})^{\frac{\delta-\alpha_0\lambda}{\alpha_0\lambda+\beta_0\mu}}B^{\frac{\lambda}{2}}.
\end{aligned} \tag{3.12}$$

(3.2) yields the following (3.13) by Löwner-Heinz theorem:

$$A^{v} \geq (A^{\frac{\mu}{2}}B^{\lambda}A^{\frac{\mu}{2}})^{\frac{\beta_0 v}{\alpha_0\lambda+\beta_0\mu}} \quad \text{for } \lambda \geq 1, \mu \geq 1 \text{ and any } v \text{ such that } \mu \geq v \geq 0. \tag{3.13}$$

Define $h(\mu)=(B^{\frac{\lambda}{2}}A^{\mu}B^{\frac{\lambda}{2}})^{\frac{\delta-\alpha_0\lambda}{\alpha_0\lambda+\beta_0\mu}}$. Then $f(\lambda,\mu)=B^{\frac{\lambda}{2}}h(\mu)B^{\frac{\lambda}{2}}$ by (3.12), and we have

$$\begin{aligned}
h(\mu) &=(B^{\frac{\lambda}{2}}A^{\mu}B^{\frac{\lambda}{2}})^{\frac{\delta-\alpha_0\lambda}{\alpha_0\lambda+\beta_0\mu}}\\
&=\{(B^{\frac{\lambda}{2}}A^{\mu}B^{\frac{\lambda}{2}})^{\frac{\alpha_0\lambda+\beta_0\mu+\beta_0 v}{\alpha_0\lambda+\beta_0\mu}}\}^{\frac{\delta-\alpha_0\lambda}{\alpha_0\lambda+\beta_0\mu+\beta_0 v}}\\
&=\{B^{\frac{\lambda}{2}}A^{\frac{\mu}{2}}(A^{\frac{\mu}{2}}B^{\lambda}A^{\frac{\mu}{2}})^{\frac{\beta_0 v}{\alpha_0\lambda+\beta_0\mu}}A^{\frac{\mu}{2}}B^{\frac{\lambda}{2}}\}^{\frac{\delta-\alpha_0\lambda}{\alpha_0\lambda+\beta_0\mu+\beta_0 v}} \qquad \text{by Lemma F}\\
&\geq(B^{\frac{\lambda}{2}}A^{\frac{\mu}{2}}A^{v}A^{\frac{\mu}{2}}B^{\frac{\lambda}{2}})^{\frac{\delta-\alpha_0\lambda}{\alpha_0\lambda+\beta_0\mu+\beta_0 v}}\\
&=(B^{\frac{\lambda}{2}}A^{\mu+v}B^{\frac{\lambda}{2}})^{\frac{\delta-\alpha_0\lambda}{\alpha_0\lambda+\beta_0(\mu+v)}}=h(\mu+v).
\end{aligned}$$

The last inequality holds by (3.13) and Löwner-Heinz theorem since $\frac{\delta-\alpha_0\lambda}{\alpha_0\lambda+\beta_0\mu+\beta_0 v}\in[-1,0]$ by the condition (3.10), and taking inverses. Hence $f(\lambda,\mu)=B^{\frac{\lambda}{2}}h(\mu)B^{\frac{\lambda}{2}}$ is decreasing for $\mu\geq 1$.

Proof of (ii). We recall the following condition on δ,α_0,β_0 and μ in (ii):

$$\text{for any fixed } \delta\leq\alpha_0 \text{ and } \mu\geq 1 \text{ such that } \beta_0\mu\geq-\delta. \tag{3.14}$$

(3.1) is equivalent to the following (3.15):

$$B^{-1}\geq(B^{\frac{-1}{2}}A^{-1}B^{\frac{-1}{2}})^{\frac{\alpha_0}{\alpha_0+\beta_0}} \quad \text{for fixed } \alpha_0\geq 0 \text{ and } \beta_0\geq 0 \text{ with } \alpha_0+\beta_0>0 \tag{3.15}$$

by Lemma F and taking inverses of both sides. We recall that (3.15) just corresponds to (3.1). And by Lemma F,

$$\begin{aligned}
f(\lambda,\mu)&=A^{\frac{-\mu}{2}}(A^{\frac{\mu}{2}}B^{\lambda}A^{\frac{\mu}{2}})^{\frac{\delta+\beta_0\mu}{\alpha_0\lambda+\beta_0\mu}}A^{\frac{-\mu}{2}}\\
&=(B^{-1})^{\frac{-\lambda}{2}}\{(B^{-1})^{\frac{\lambda}{2}}(A^{-1})^{\mu}(B^{-1})^{\frac{\lambda}{2}}\}^{\frac{-\delta+\alpha_0\lambda}{\beta_0\mu+\alpha_0\lambda}}(B^{-1})^{\frac{-\lambda}{2}}.
\end{aligned} \tag{3.16}$$

By applying (i), for any fixed $-\delta\geq-\alpha_0$, $f(\lambda,\mu)$ is decreasing for $\lambda\geq 1$ and $\mu\geq 1$ under the condition (3.14) by (3.15) and (3.16), so the proof of (ii) is complete. The equivalence relation between (i) and (ii) is obvious by scrutinizing the proof of (i) and (ii).

Consequently we have finished the proof of Theorem 1. □

Proof of Theorem 2. We may assume that A and B are both invertible in the proof. In the case $t=0$, the result follows by [8, Theorem 3], so we have only to consider the case $p\geq t>0$.

Proof of (i). Put $X=A^{\frac{-t}{2}}B^{p}A^{\frac{-t}{2}}$. Then X is positive invertible and $A\geq(A^{\frac{t}{2}}XA^{\frac{t}{2}})^{\frac{1}{p}}$ by the hypothesis $A\geq B\geq 0$. Put $\beta_0=t\in(0,1]$ and $\alpha_0=p-t\geq 0$. Then $A\geq(A^{\frac{t}{2}}XA^{\frac{t}{2}})^{\frac{1}{\alpha_0+\beta_0}}$, so that

$$A^{t}\geq(A^{\frac{t}{2}}XA^{\frac{t}{2}})^{\frac{\beta_0}{\alpha_0+\beta_0}}$$

holds by Löwner-Heinz theorem. Put $r=\mu\beta_0=\mu t\geq t$ and $\delta=q-t$. And define $f(s,\mu)=A^{\frac{-\mu t}{2}}(A^{\frac{\mu t}{2}}X^{s}A^{\frac{\mu t}{2}})^{\frac{\delta+\mu t}{\alpha_0 s+\mu t}}A^{\frac{-\mu t}{2}}$, then we have

$$\begin{aligned}
f(s,\mu)&=A^{\frac{-\mu t}{2}}(A^{\frac{\mu t}{2}}X^{s}A^{\frac{\mu t}{2}})^{\frac{\delta+\mu t}{\alpha_0 s+\mu t}}A^{\frac{-\mu t}{2}}\\
&=A^{\frac{-r}{2}}\{A^{\frac{r}{2}}(A^{\frac{-t}{2}}B^{p}A^{\frac{-t}{2}})^{s}A^{\frac{r}{2}}\}^{\frac{q-t+r}{(p-t)s+r}}A^{\frac{-r}{2}}\\
&=G_{p,q,t}(A,B,r,s).
\end{aligned} \tag{3.17}$$

By using (i) of Theorem 1 since $\delta \geq -\beta_0$ holds by $q \geq 0$, $f(s,\mu)$ is decreasing for $\mu \geq 1$ and $s \geq 1$ such that $\alpha_0 s \geq \delta$, so that $G_{p,q,t}(A,B,r,s)$ is decreasing for $r \geq t$ and $s \geq 1$ such that $(p-t)s \geq q-t$. Hence the proof of (i) is complete.
Proof of (ii). The condition $p \geq q$ and $r \geq t-q$ in (ii) satisfy $\delta \leq \alpha_0$ and $\beta_0\mu \geq -\delta$ in the conditions of (ii) in Theorem 1, so that $G_{p,q,t}(A,B,r,s)$ is decreasing for $s \geq 1$ and $r \geq \max\{t, t-q\}$ by (ii) of Theorem 1 and (3.17). The equivalence relation between (i) and (ii) follows by Theorem 1.

Hence the proof of Theorem 2 is complete. $\square$

Proof of Corollary 3. We recall the following (3.18) in [4][9], which is an extension of [1]:

$$A \gg B \text{ holds if and only if } A^r \geq (A^{\frac{r}{2}}B^pA^{\frac{r}{2}})^{\frac{r}{p+r}} \text{ for all } p \geq 0 \text{ and } r \geq 0. \tag{3.18}$$

(i) $\Longrightarrow$ (ii). Assume (i). As (3.18) holds, by (i) of Theorem 1, for any fixed $q \geq 0$

$$f(\lambda,\mu) = A^{\frac{-r\mu}{2}}(A^{\frac{r\mu}{2}}B^{p\lambda}A^{\frac{r\mu}{2}})^{\frac{q+r\mu}{p\lambda+r\mu}}A^{\frac{-r\mu}{2}}$$

is decreasing for $\mu \geq 1$ and $\lambda \geq 1$ such that $p\lambda \geq q$, that is, for any fixed $q \geq 0$, $F_q(p,r)$ is decreasing for $p \geq q$ and $r \geq 0$.
(i) $\Longrightarrow$ (iii). In the same way as the proof (i) $\Longrightarrow$ (ii) by using (ii) of Theorem 1.
(ii) $\Longrightarrow$ (i). Assume that $F_q(p,r)$ is decreasing for $r \geq 0$. Then $F_0(p,0) \geq F_0(p,r)$ holds, that is, $I \geq A^{\frac{-r}{2}}(A^{\frac{r}{2}}B^pA^{\frac{r}{2}})^{\frac{r}{p+r}}A^{\frac{-r}{2}}$, so that $A^r \geq (A^{\frac{r}{2}}B^pA^{\frac{r}{2}})^{\frac{r}{p+r}}$ for all $p \geq 0$ and $r \geq 0$, which is equivalent to $A \gg B$ by (3.18).
(iii) $\Longrightarrow$ (i). In the same way as the proof (ii) $\Longrightarrow$ (i).

Hence the proof of Corollary 3 is complete. $\square$

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PARAMETRIZED FURUTA INEQUALITY

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1. Introduction. Ten years ago, Furuta [5] established an interesting operator inequality which is an extension of the Löwner-Heinz inequality and is now called the Furuta inequality. Afterward, Furuta [7] proposed the grand Furuta inequality which interpolates the Furuta inequality and the Ando-Hiai inequality [1]. It can be expressed as follows:

The grand Furuta inequality: *If $A \geq B \geq 0$ and A is invertible, then for each $p \geq 1$ and $0 \leq t \leq 1$,*

$$A^{-r+t} \sharp_{\frac{1-t+r}{(p-t)s+r}} (A^t \natural_s B^p) \leq A \tag{1}$$

holds for $r \geq t$ and $s \geq 1$.
Here we used the notaions $\sharp_\alpha$ and $\natural_\alpha$ defined for positive operators A and B by

$$A \natural_\alpha B = A^{\frac{1}{2}}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^\alpha A^{\frac{1}{2}}, \quad for\ \alpha \in \mathbf{R}$$

and $\sharp_\alpha = \natural_\alpha$ when $\alpha \in [0,1]$. Note that $\sharp_\alpha$ is an operator mean in the sense of Kubo-Ando [15] which corresponds to the operator monotone function x^α in the Löwner theory.

In [4,cf.13,14], we tried to approach mean theoretically to the grand Furuta inequality where we showed the following inequality which seems to be fundamental in our arguments of the grand Furuta inequality.

Theorem A. *If $A \geq B > 0$, then for $t \in [0,1]$ and $p \geq 1$,*

$$(A^t \natural_{\frac{\beta-t}{p-t}} B^p)^{\frac{1}{\beta}} \leq B \leq A \tag{2}$$

holds for $\beta \geq p$ and $p \neq t$.

Recently, Furuta-Yamazaki-Yanagida [9,10] proved the following order preserving operator inequality which is an extended form of the grand Furuta inequality; the proof is based on elementary calculations and clarifies the importance of the Furuta inequality.

Theorem B. *If $A \geq B \geq 0$ and A is invertible, then for $0 \leq t \leq \delta \leq 1$ and $\delta \leq p$,*

$$A^{-r+t} \sharp_{\frac{\delta-t+r}{(p-t)s+r}} (A^t \natural_s B^p) \leq A^\delta \tag{3}$$

holds for $r \geq t$ and $s \geq 1$.

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The original proof of this theorem in [7] is somewhat complicated and hard. The proof given in [9] is simplified by using the Furuta inequality. In this note, we show Theorem B is more extensible, and Theorem A is essential as well as in the preceding proof of [4]. In other words, Theorem A might be fundamental among such inequalities of grand Furuta type.

2. Preliminary. First of all, we recall the Furuta inequality for convenience. Throughout this note, a capital letter means a bounded linear operator on a Hilbert space H. An operator A is said to be positive (in symbol: $A \geq 0$) if $(Ax, x) \geq 0$ for all $x \in H$, and also an operator A is strictly positive (in symbol: $A > 0$) if A is positive and invertible.

The original form of the Furuta inequality [5] given by Furuta himself is the following(cf.[6]).

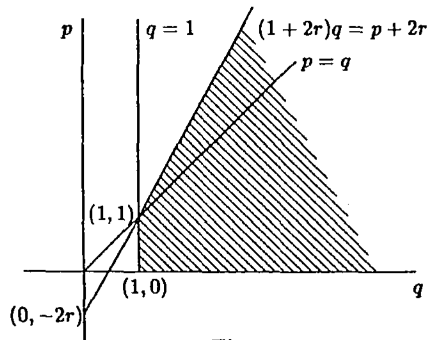
Furuta inequality: *If $A \geq B \geq 0$, then for each $r \geq 0$,*

$$(A^r A^p A^r)^{\frac{1}{q}} \geq (A^r B^p A^r)^{\frac{1}{q}} \tag{4}$$

and

$$(B^r A^p B^r)^{\frac{1}{q}} \geq (B^r B^p B^r)^{\frac{1}{q}} \tag{4'}$$

holds for p and q such that $p \geq 0$ and $q \geq 1$ with $(1+2r)q \geq p+2r$.



Figure

As known in [2,3,11,12,13,14], it can be represented in terms of the operator mean $\sharp_\alpha$ as follows:

Satellite theorem of the Furuta inequality: *If $A \geq B \geq 0$, then*

$$A^u \sharp_{\frac{1-u}{p-u}} B^p \leq B \leq A \leq B^u \sharp_{\frac{1-u}{p-u}} A^p \tag{5}$$

for all $p \geq 1$ and $u \leq 0$.

From this point of view, we recently obtained the following extension of a satellite theorem of the Furuta inequality [14,cf.3].

Theorem C. *If $A \geq B > 0$, then for each $p \geq 0$ and $u \leq 0$ with $p \neq u$,*

$$A^u \sharp_{\frac{\delta-u}{p-u}} B^p \leq B^\delta \tag{6}$$

holds for $\delta \in [0, p]$.
In this theorem, the Furuta inequality or the satellite theorem is corresponding to the case of $\delta = 1$.

We note that Furuta [8] recently proposed its generalization of grand Furuta type.

Cocluding this section, we present a typical role of Theorem A; the following initial case of the grand Furuta inequality has a simple proof based on Theorem A and the Furuta inequality as follows:

Let $B_1 = (A^t \natural_{\frac{\beta-t}{p-t}} B^p)^{\frac{1}{\beta}}$, then $A \geq B \geq B_1$ by Theorem A. Using the Furuta inequality or (5), we have

$$A^u \sharp_{\frac{1-u}{\beta-u}} B_1^\beta \leq B_1 \leq B \leq A$$

for $u \leq 0$. Replacing $\frac{\beta-t}{p-t} = s$ and $u = -r+t$ $(r \geq t)$, (1) is obtained.

3. Results. To make sure, we give a proof of Theorem A which is different from the one given in [4].

Proof of Theorem A. First of all, we note the following formula which is easily obtained by the definition:

$$A \natural_\alpha B = A(A^{-1}\natural_{-\alpha} B^{-1})A \quad for\ A,\ B > 0.$$

In the first case of $1 \leq \frac{\beta-t}{p-t} \leq 2$;

$$\begin{aligned} A^t \natural_{\frac{\beta-t}{p-t}} B^p &= B^p \natural_{1-\frac{\beta-t}{p-t}} A^t = B^p(B^{-p} \sharp_{\frac{\beta-t}{p-t}-1} A^{-t})B^p \\ &\leq B^p(B^{-p} \sharp_{\frac{\beta-p}{p-t}} B^{-t})B^p = B^{2p+((p-t)\frac{\delta-p}{p-t}-p)} = B^\beta. \end{aligned}$$

Hence we have $(A^t \natural_{\frac{\beta-t}{p-t}} B^p)^{\frac{1}{\beta}} \leq B \leq A$.

As the second case, we choose β_1; $1 \leq \frac{\beta_1-t}{\beta-t} \leq 2$. Then for A and $B_1 = (A^t \natural_{\frac{\beta-t}{p-t}} B^p)^{\frac{1}{\beta}}$, we can repeat similar calculations as follows;

$$\begin{aligned} A^t \natural_{\frac{\beta_1-t}{p-t}} B^p &= A^t \natural_{\frac{\beta_1-t}{\beta-t}} (A^t \natural_{\frac{\beta-t}{p-t}} B^p) = A^t \natural_{\frac{\beta_1-t}{\beta-t}} B_1^\beta \\ &= B_1^\beta \natural_{1-\frac{\beta_1-t}{\beta-t}} A^t = B_1^\beta(B_1^{-\beta} \sharp_{\frac{\beta_1-t}{\beta-t}-1} A^{-t})B_1^\beta \\ &\leq B_1^\beta(B_1^{-\beta} \sharp_{\frac{\beta_1-\beta}{\beta-t}} B_1^{-t})B_1^\beta = B_1^{\beta_1} = (A^t \natural_{\frac{\beta-t}{p-t}} B^p)^{\frac{\beta_1}{\beta}}. \end{aligned}$$

So we have $(A^t \natural_{\frac{\beta_1-t}{p-t}} B^p)^{\frac{1}{\beta_1}} \leq (A^t \natural_{\frac{\beta-t}{p-t}} B^p)^{\frac{1}{\beta}} \leq B \leq A$ by the Löwner-Heinz inequality.
The third case, we choose β_2; $1 \leq \frac{\beta_2-t}{\beta_1-t} \leq 2$, and repeating the above method, we can attain the conclusion.

Remark. By this proof, we can see at the same time the operator function $f(\beta) = (A^t \natural_{\frac{\beta-t}{p-t}} B^p)^{\frac{1}{\beta}}$ being monotone decreasing for $\beta \geq p$.

Theorem B is not only an extension but also a good explanations of the grand Furuta inequality. Based on it, we give an extension of Theorem B. As seen in Theorem 1 below, Theorem A looks like a starting block and the Furuta inequality leads us to the goal.

Theorem 1. *If* $A \geq B > 0$, *then*

$$A^u \sharp_{\frac{\delta-u}{\beta-u}} (A^t \natural_{\frac{\beta-t}{p-t}} B^p) \leq (A^t\natural_{\frac{\beta-t}{p-t}} B^p)^{\frac{\delta}{\beta}} \leq B^\delta \leq A^\delta \tag{7}$$

holds for $0 \leq t \leq \delta \leq 1$, $\delta \leq p \leq \beta, p \neq t$ *and* $u \leq 0$.

At first, we give a direct proof of this theorem by using Theorem A.

Proof of Theorem 1. Let $A_0 = A^\delta, B_0 = B^\delta, t_0 = \frac{t}{\delta}, p_0 = \frac{p}{\delta}$ *and* $\beta_0 = \frac{\beta}{\delta}$, then $A_0 \geq B_0$ and $0 \leq t_0 \leq 1 \leq p_0 \leq \beta_0$. Using Theorem A, we have

$$(A_0^{t_0} \natural_{\frac{\beta_0 - t_0}{p_0 - t_0}} B_0^{p_0})^{\frac{1}{\beta_0}} \leq B_0 \leq A_0,$$

that is,

$$(A^t \natural_{\frac{\beta - t}{p-t}} B^p)^{\frac{\delta}{\beta}} \leq B^\delta \leq A^\delta.$$

By putting $A_1 = A^\delta \geq B_1 = (A^t \natural_{\frac{\beta-t}{p-t}} B^p)^{\frac{\delta}{\beta}}$, $u_1 = \frac{u}{\delta} \leq 0$ and $p_1 = \frac{\beta}{\delta} \geq 1$, the Furuta inequality or (5) says

$$A_1^{u_1} \sharp_{\frac{1-u_1}{p_1-u_1}} B_1^{p_1} \leq B_1 \leq A_1.$$

Melting this, we have the conclusion.

The proof of Theorem 1 is very short but Theorem 1 is an extension of Theorem B. As a matter of fact, it is rephrased by the same formula as that of Theorem B:

Theorem 1'. *If $A \geq B \geq 0$ and A is invertible, then for $0 \leq t \leq \delta \leq 1$ and $\delta \leq p$,*

$$A^{-r+t} \sharp_{\frac{\delta - t + r}{(p-t)s+r}} (A^t \natural_s B^p) \leq (A^t \natural_s B^p)^{\frac{\delta}{(p-t)s+t}} \leq B^\delta \leq A^\delta \tag{3'}$$

holds for $r \geq t$ and $s \geq 1$.

In the remainder, we discuss some results around Theorem 1. In the conditions of Theorem 1, we can loosen δ more freely as follows:

Theorem 2. *If $A \geq B > 0$, then*

$$A^u \sharp_{\frac{\delta-u}{\beta-u}} (A^t \natural_{\frac{\beta-t}{p-t}} B^p) \leq (A^t \natural_{\frac{\beta-t}{p-t}} B^p)^{\frac{\delta}{\beta}} \tag{8}$$

for $0 \leq t \leq 1 \leq p \leq \beta$, $p \neq t$, $u \leq 0$ and $\delta \in [0, \beta]$.

Proof. By Theorem A, we have $(A^t \natural_{\frac{\beta-t}{p-t}} B^p)^{\frac{1}{\beta}} \leq B \leq A$. Applying Theorem C to A and $B_1 = (A^t \natural_{\frac{\beta-t}{p-t}} B^p)^{\frac{1}{\beta}}$, we have $A^u \sharp_{\frac{\delta-u}{\beta-u}} B_1^\beta \leq B_1^\delta$, that is,

$$A^u \sharp_{\frac{\delta-u}{\beta-u}} (A^t \natural_{\frac{\beta-t}{p-t}} B^p) \leq (A^t \natural_{\frac{\beta-t}{p-t}} B^p)^{\frac{\delta}{\beta}}.$$

In the preceding note [14], we have shown the following as a little extension of the grand Furuta inequality which is now a corollary of this theorem.

Corollary 3. *If $A \geq B > 0$, then*

$$A^u \sharp_{\frac{\delta-u}{\beta-u}} (A^t \natural_{\frac{\beta-t}{p-t}} B^p) \leq (A^t \natural_{\frac{\beta-t}{p-t}} B^p)^{\frac{\delta}{\beta}} \leq B^\delta \leq A^\delta \tag{9}$$

for $0 \leq t \leq 1 \leq p \leq \beta$, $p \neq t$, $u \leq 0$ and $\delta \in [0, 1]$.

If we choose $\delta = 1$, then we have the following which is an extennsion of (1).

Corollary 4. *If $A \geq B > 0$, then*

$$A^u \,\sharp_{\frac{1-u}{\beta-u}}\, (A^t \,\natural_{\frac{\beta-t}{p-t}}\, B^p) \leq (A^t \,\natural_{\frac{\beta-t}{p-t}}\, B^p)^{\frac{1}{\beta}} \leq B \leq A \tag{10}$$

for $0 \leq t \leq 1 \leq p \leq \beta,\ p \neq t,\ u \leq 0$.

Concluding this note, we return to Theorem 1; we give another proof of Theorem 1 by using Corollary 4:

If $0 < \delta \leq 1$, then $A^\delta \geq B^\delta$ and $0 \leq \frac{t}{\delta} \leq 1 \leq \frac{p}{\delta} \leq \frac{\beta}{\delta}, \frac{u}{\delta} \leq 0$. Let $A_1 = A^\delta, B_1 = B^\delta, t_1 = \frac{t}{\delta}, p_1 = \frac{p}{\delta}, \beta_1 = \frac{\beta}{\delta}$ and $u_1 = \frac{u}{\delta}$, then they satisfy the conditions of Corollary 4. So we have

$$A_1^{u_1} \,\sharp_{\frac{1-u_1}{\beta_1-u_1}}\, (A_1^{t_1} \,\natural_{\frac{\beta_1-t_1}{p_1-t_1}}\, B_1^{p_1}) \leq (A_1^{t_1} \,\natural_{\frac{\beta_1-t_1}{p_1-t_1}}\, B_1^{p_1})^{\frac{1}{\beta_1}} \leq B_1 \leq A_1.$$

Rewriting this, we can obtain Theorem 1.

Very recently, Furuta has pointed out in our private communication that Theorem 2 is improvable as follows:

Theorem 2'. *If $A \geq B > 0$, then*

$$A^u \,\sharp_{\frac{\delta-u}{\beta-u}}\, (A^t \,\natural_{\frac{\beta-t}{p-t}}\, B^p) \leq (A^t \,\natural_{\frac{\beta-t}{p-t}}\, B^p)^{\frac{\delta}{\beta}} \tag{8}$$

for $0 \leq t < p \leq \beta,\ u \leq 0$ *and* $\delta \in [0, \beta]$.

Proof. We have only to show the case of $0 \leq t < p \leq 1$. In this case, we can apply Theorem A to $A^p \geq B^p \geq 0$ and have

$$(A^t \,\natural_{\frac{\beta-t}{p-t}}\, B^p)^{\frac{p}{\beta}} = ((A^p)^{\frac{t}{p}} \,\natural_{\frac{\frac{\beta}{p}-\frac{t}{p}}{1-\frac{t}{p}}}\, B^p)^{\frac{p}{\beta}} \leq B^p \leq A^p.$$

Let $A_1 = A^p$, $B_1 = (A^t \,\natural_{\frac{\beta-t}{p-t}}\, B^p)^{\frac{p}{\beta}}$ and $u_1 = \frac{u}{p}$, $\delta_1 = \frac{\delta}{p}$, $p_1 = \frac{\beta}{p}$. Then Theorem C says

$$A_1^{u_1} \,\sharp_{\frac{\delta_1-u_1}{p_1-u_1}}\, B_1^{p_1} \leq B_1^{\delta_1}.$$

Rewriting this, we have the conclusion.

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THE WIELANDT THEOREM:
SIMPLE PROOFS AND GENERALIZATIONS

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1. Introduction.

This is a joint work with Y.Katayama and R.Nakamoto.

Throughout this note, an operator means a bounded linear operator acting on a Hilbert space H. If H is finite dimensional, then an operator is a matrix whose entries are complex numbers. An operator A is positive if $(Ax,x) \geq 0$ for all $x \in H$, and the positivity induces the order $A \geq B$ by $A - B \geq 0$.

In our recent works [2,3,4,6], we discuss the Kantorovich inequality: If a positive operator A on H satisfies $0 < m \leq A \leq M$ for some $m < M$, then

$$(Ax,x)(A^{-1}x,x) \leq \frac{(M+m)^2}{4Mm} \tag{0}$$

for all unit vectors $x \in H$. Related to this, we pay attention to the Wielandt theorem, which is regarded as an improvement of the Cauchy-Schwarz inequality, see [5; 7.4.34].

The Wielandt theorem. *If $0 < m \leq A \leq M$, then*

$$|(Ay,x)|^2 \leq \left(\frac{M-m}{M+m}\right)^2 (Ax,x)(Ay,y) \tag{1}$$

for every orthogonal pair x and y.

Following after Turing [7], the condition number of an invertible operator A is defined by $\kappa(A) = \|A\|\|A^{-1}\|$, so that $\kappa = \frac{M}{m}$ might be a generalized condition number of a positive operators with $0 < m \leq A \leq M$, cf. [4]. Putting the angle θ by $\cot\frac{\theta}{2} = \sqrt{\kappa}$, we have $\cos\theta = \frac{\kappa-1}{\kappa+1}$ and so (1) is rephrased as

$$\frac{|(Bx,By)|}{\|Bx\|\|By\|} \leq \cos\theta \tag{1'}$$

for every orthogonal pair x and y, where $B = A^{\frac{1}{2}}$ is the square root of A, see [5, 7.4.32].

On the other hand, Bauer and Householder generalized the Wielandt theorem to the case where vectors x and y are not orthogonal; the following theorem is essentially due to them [1; Theorem II]:

The Bauer-Householder theorem. *If $0 < m \leq A \leq M$ and x, y are unit vectors, then*

$$|(Ay,x)|^2 \leq \left(\frac{\tilde{M}-\tilde{m}}{\tilde{M}+\tilde{m}}\right)^2 (Ax,x)(Ay,y), \tag{2}$$

where $\tilde{M} = M(1 + |(x, y)|)$ *and* $\tilde{m} = m(1 - |(x, y)|)$.

2. Simple proofs. Inspired by a proof of the Cauchy-Schwarz inequality with the use of the discriminant, we have the first proof of the Wielandt theorem:

Proof 1. Suppose that $\|x\| = \|y\| = 1$ and $(x, y) = 0$. By the assumption $m \le A \le M$, we have

$$m\|x + zy\|^2 \le (A(x + zy), x + zy) \le M\|x + zy\|^2 \tag{3}$$

for all $z \in \mathbb{C}$. We may assume that $(Ay, x) \ge 0$ and so (3) holds for all $z \in \mathbb{R}$. Then it follows that

$$z^2((Ay, y) - m) + 2z(Ay, x) + (Ax, x) - m \ge 0 \tag{4}$$

and

$$z^2(M - (Ay, y)) + 2z(Ay, x) + M - (Ax, x) \ge 0 \tag{5}$$

for all $z \in \mathbb{R}$. Calculating (4) $\times$ M + (5) $\times$ m, we have

$$z^2(M - m)(Ay, y) + 2z(M + m)(Ay, x) + (M - m)(Ax, x) \ge 0 \tag{6}$$

for all $z \in \mathbb{R}$, so that

$$(M + m)^2(Ay, x)^2 \le (M - m)^2(Ax, x)(Ay, y)$$

by taking the discriminant of (6), and as desired.

The second one is along with the proof stated in Horn-Johnson's text [5; 7.4.26], but the Poincare separation theorem is cleared by an operator theoretic method:

Proof 2. Also suppose that $\|x\| = \|y\| = 1$ and $(x, y) = 0$. Put the 2×2 matrix

$$C = \begin{pmatrix} (Ax, x) & (Ay, x) \\ \overline{(Ay, x)} & (Ay, y) \end{pmatrix}.$$

Then $m \le C \le M$ because $\|\binom{\alpha}{\beta}\| = \|\alpha x + \beta y\| = 1$ and

$$\left(C\begin{pmatrix}\alpha \\ \beta\end{pmatrix}, \begin{pmatrix}\alpha \\ \beta\end{pmatrix}\right) = (A(\alpha x + \beta y), \alpha x + \beta y) \in [m, M].$$

Hence the spectrum $\{a, b\}$ of C is contained in $[m, M]$.

The second half is the same as the proof of [5; 7.4.26]; we sketch it for the sake of completeness. Since

$$1 - \frac{|(Ay, x)|^2}{(Ax, x)(Ay, y)} = \frac{4 \det C}{(trC)^2 - ((Ax, x) - (Ay, y))^2} \ge \frac{4 \det C}{(trC)^2} = \frac{4ab}{(a + b)^2},$$

we have

$$\frac{|(Ay, x)|^2}{(Ax, x)(Ay, y)} \le 1 - \frac{4ab}{(a + b)^2} = \left(\frac{1 - \frac{b}{a}}{1 + \frac{b}{a}}\right)^2 \le \left(\frac{1 - \frac{M}{m}}{1 + \frac{M}{m}}\right)^2 = \left(\frac{M - m}{M + m}\right)^2$$

by the monotone increase of the function $\frac{t-1}{t+1}$.

The second proof gives us a simple proof of the Bauer-Householder theorem, which is due to a kind suggestion of the referee. For this, the following lemma is needed; it is an extension of the first half in Proof 2.

Lemma 1. *If* $0 < m \leq A \leq M$ *and*

$$C = \begin{pmatrix} (Ax, x) & (Ax, y) \\ (y, Ax) & (Ay, y) \end{pmatrix}$$

for given unit vectors x *and* y*, then the spectrum* $\sigma(C)$ *of* C *is contained in the interval* $[(1-t)m, (1+t)M]$*, where* $t = |(x, y)|$.

Proof. Let $\overline{W}(A)$ be the closed numerical range of an operator A on a Hilbert space, i.e., the closure of $W(A) = \{(Ax, x); \|x\| = 1, x \in H\}$. We first prove that

$$\overline{W}(X^*AX) \subseteq \overline{W}(A)\overline{W}(X^*X)$$

for an operator X of a Hilbert space K into H. If $Xz = 0$, then $0 = (X^*AXz, z) \in W(A)W(X^*X)$. If $Xz \neq 0$ and $\|z\| = 1$, then

$$(X^*AXz, z) = \|Xz\|^2 (A\frac{Xz}{\|Xz\|}, \frac{Xz}{\|Xz\|}) \in W(A)W(X^*X).$$

We here take $X = [x, y]$. Then $C = X^*AX$ and $\overline{W}(X^*X) = \sigma(X^*X) = [1-t, 1+t]$ because

$$X^*X = \begin{pmatrix} (x, x) & (x, y) \\ (y, x) & (y, y) \end{pmatrix} = \begin{pmatrix} 1 & (x, y) \\ (y, x) & 1 \end{pmatrix}.$$

Since $\sigma(C) \subseteq \overline{W}(C)$ in general, we have

$$\sigma(C) \subseteq \overline{W}(X^*AX) \subseteq \overline{W}(A)\overline{W}(X^*X) \subseteq [m, M][1-t, 1+t] = [(1-t)m, (1+t)M].$$

By using Lemma 1, we can give a simple proof to the Bauer-Householder theorem along with Proof 2:

Proof of the Bauer-Householder theorem. Lemma 1 is corresponding to the first half of Proof 2. As in the proof 2, we have

$$\frac{|(Ay, x)|^2}{(Ax, x)(Ay, y)} \leq 1 - \frac{4ab}{(a+b)^2} = \left(\frac{1 - \frac{b}{a}}{1 + \frac{b}{a}}\right)^2,$$

where $\sigma(C) = \{a, b\}$. Since $\sigma(C) \subseteq [(1-t)m, (1+t)M]$, it follows that

$$\frac{|(Ay, x)|^2}{(Ax, x)(Ay, y)} \leq \left(\frac{\tilde{M} - \tilde{m}}{\tilde{M} + \tilde{m}}\right)^2,$$

where $\tilde{m} = (1-t)m$ and $\tilde{M} = (1+t)M$.

Remark. For a given invertible operator B, if we apply (2) to $A = B^*B$, then we have the original form of [1; Theorem II].

3. Generalizations. Based on simple proofs in the preceding section, we can generalize the Wielandt theorem without assumption on the orthogonality of vectors. The following one corresponds to Proof 1, in which $|(Ay, x)|$ can be estimated by adding to the term $\frac{2Mm}{M+m}|(x, y)|$; we remark that its coefficient is the harmonic mean of M and m.

Theorem 1. *If* $0 < m \leq A \leq M$, *then*

$$|(Ay,x)| \leq \frac{M-m}{M+m}(Ax,x)^{\frac{1}{2}}(Ay,y)^{\frac{1}{2}} + \frac{2Mm}{M+m}|(x,y)|$$

for all $x, y \in H$.

Corresponding to the second proof of the Wielandt theorem, we have another generalization of it, whose point is the following lemma:

Lemma 2. *Suppose that* $0 < m \leq A \leq M$. *For each unit vectors* $x, y \in H$, *take* $t \geq 0$ *satisfying*

$$|(Ay,x) - t(y,x)| \leq |(Ay,x) - k(y,x)| \quad \textit{for} \quad k = m, M$$

and put

$$C_t = \begin{pmatrix} (Ax,x) & (Ay,x) - t(y,x) \\ \overline{(Ay,x)} - t\overline{(y,x)} & (Ay,y) \end{pmatrix}.$$

Then the spectrum $\sigma(C_t)$ *is contained in* $[m, M]$.

Theorem 3. *Notation as in above. Then*

$$(10) \qquad |(Ay,x)| \leq \frac{M-m}{M+m}(Ax,x)^{\frac{1}{2}}(Ay,y)^{\frac{1}{2}} + t|(x,y)|.$$

Corollary 4. *If* $0 < m \leq A \leq M$, *then the Wielandt inequality*

$$(1) \qquad |(Ay,x)|^2 \leq \left(\frac{M-m}{M+m}\right)^2 (Ax,x)(Ay,y)$$

holds for all $x, y \in H$ *satisfying* $Re(Ay,x)\overline{(y,x)} \leq 0$.

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Cellular Automata in Function Spaces

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1. Introduction

A cellular automaton consists of d-dimensional lattice ($\boldsymbol{Z}^d, d \in \boldsymbol{N}$), and each site takes a state, one of a finite set of possible values. The value of each site evolves in discrete time steps and it is determined by the previous values of a neighborhood of sites around it.

Let $\mathcal{P}^d$ be the set of all configurations : $\boldsymbol{Z}^d \to \boldsymbol{Z}/p$. A map L: $\mathcal{P}^d \to \mathcal{P}^d$ is a *transition rule* if (1) $L(0) = 0$; and (2) there exist $v_1, \cdots, v_m \in \boldsymbol{Z}^d$ and a map $f : (\boldsymbol{Z}/p)^m \to \boldsymbol{Z}/p$ such that

$$(La)(x) = f(a(x+v_1), \cdots, a(x+v_m)) \text{ for all } x \in \boldsymbol{Z}^d, a \in \mathcal{P}^d. \tag{1.1}$$

To consider space-time patterns of cellular automata, we shall study the sequence a, La, $L^2a = L(La)$, $L^3a, \ldots$. If a is any finite nonzero configuration, for any k, putting $a, La, L^2a \cdots L^ka$ on (d+1)-dimensional lattice in oder, contracting by $1/2^k$, one obtains $G_L^k a$ as a subset of $\boldsymbol{R}^d \times [0,1]$. S. Willson [3] studied when L is linear modulo 2 and showed there exists a stable limit set of $G_L^k a$ as $k \to \infty$ and the limit set is independent of an initial configuration a, if a is finite and nonzero. S. Takahashi investigated the self-similarity for linear cellular automata with $p \geq 2$ in [2], where the limit set of non-zero states or each k-state $(1 \leq k \leq p-1)$ is considered as a subset of $\boldsymbol{R}^d$. The set theory plays an important role in [2] and [3] . However, when p is greater than 3, it is useful to consider a finite-valued function and it may be helpful to use the operator theory.

In this paper, we discuss the behavior of cellular automata by using the operator theory. In section 2, we introduce the product space $\prod E_k$ and the operator $\bar{F}_L$ on it corresponding to L. We consider the behavior of $\bar{F}_L^k$ as $k \to \infty$. Futhermore we define a quotient space $\tilde{E} = \prod E_k / \sim$ and the operator $\tilde{F}_L$ on it and investigate a condition that a $\tilde{F}_L$−invariant set belongs to a certain space. In section 3, we discuss the case of linear rules. In section 4, we will consider L is a non-linear rule. We show some conditions such that there exists a $\tilde{F}_L$−invariant set.

2. Operators on the space USC and their limit

We shall consider cellular automata taking the value $\boldsymbol{Z}/2$. A *configuration* a on $\boldsymbol{Z}^d$ is a map a : $\boldsymbol{Z}^d \to \boldsymbol{Z}/2$ and $\mathcal{P}^d$ is the set of all configurations on $\boldsymbol{Z}^d$. A configuration a is *finite* provided $a(v) = 1$ for only finitely many v. We define two kinds of addition: If $a, b \in \mathcal{P}^d$ we may define $a + b \in \mathcal{P}^d$ by $(a+b)(v) = a(v) + b(v)$ mod 2 for $v \in \boldsymbol{Z}^d$. If $x, v \in \boldsymbol{Z}^d$, we may define the *translate* of $a \in \mathcal{P}^d$ by v as $a\tilde{+}v$ where $(a\tilde{+}v)(x) = a(x-v)$. For $x \in \boldsymbol{Z}^d$, we define $\delta_x \in \mathcal{P}^d$ as

$$\delta_x(y) = \begin{cases} 1 & x = y \\ 0 & x \neq y \end{cases} .$$

$\mathcal{P}_+^{d+1}$ is the set of all maps $w : \boldsymbol{Z}^d \times \boldsymbol{N} \to \boldsymbol{Z}/2$, and $\mathcal{P}_{+,k}^{d+1}$ is the set of $w \in \mathcal{P}_+^{d+1}$ such that $w(x,t) = 0$ for $t \geqq k$.

$G_{L,k} : \mathcal{P}_{+,2^k}^{d+1} \to \mathcal{P}_{+,2^{k+1}}^{d+1}$ is defined by

$$G_{L,k}w(x,t) = \begin{cases} w(x,t) & 0 \leq t \leq 2^k - 1 \\ (L^{t+1-2^k} w_0)(x) & 2^k \leq t \leq 2^{k+1} - 1 \\ 0 & 2^{k+1} \leq t \end{cases} \quad \text{for } w_0(x) = w(x, 2^k - 1).$$

Let $USC(\boldsymbol{R}^d \times [0,1])$ be the set of all upper semi continuous functions $g : \boldsymbol{R}^d \times [0,1] \to \{0,1\}$. The map $\phi_k : \mathcal{P}^{d+1}_{+,2^k} \to USC(\boldsymbol{R}^d \times [0,1])$ is defined by

$$\phi_k(w)(x,t) = \inf\{\psi(x,t) | \psi \in USC(\boldsymbol{R}^d \times [0,1]), \psi(x,t) \geq w([2^k x], [2^k t])\}$$

for $w \in \mathcal{P}^{n+1}_{+,2^k}$, where $[2^k x] = ([2^k x_1], [2^k x_2], \cdots, [2^k x_n])$ for $x = (x_1, x_2, \cdots, x_n)$ and $[2^k x_j]$ means the Gauss's symbol.

$G^k_L : \mathcal{P}^d \to USC(\boldsymbol{R}^d \times [0,1])$ is defined by

$$G^k_L a = \phi_k \prod_{j=0}^{k-1} G_{L,j} a \tag{2.1}$$

for $a \in \mathcal{P}^d$.

Remark 1. If L is linear, then G^k_L is also linear.

We define $f \geq g$ for $f, g \in USC(\boldsymbol{R}^d \times [0,1])$, if $f(x) \geq g(x)$ for all $x \in \boldsymbol{R}^d \times [0,1]$. Then $USC(\boldsymbol{R}^d \times [0,1])$ is a complete lattice [1, chap. 2]. For any $\{f_n\} \subset USC(\boldsymbol{R}^d \times [0,1])$, the relation $\bigwedge_{n=1}^{\infty} \bigvee_{k \geq n} f_k \geqq \bigvee_{n=1}^{\infty} \bigwedge_{k \geq n} f_k$ holds. If they are equal we denote them both by $\lim_{n\to\infty} f_n$ in USC($\boldsymbol{R}^d \times [0,1]$).

Remark 2. The existence of $\lim_{n\to\infty} G^n_L a$ depends on L and on the initial configuration a.

In order to investigate the existence of the limit set, we shall consider a product space. Let $E_k = \phi_k(\mathcal{P}^{d+1}_{+,2^k})$, then $E_0 \subset E_1 \subset E_2 \subset \cdots \subset USC(\boldsymbol{R}^d \times [0,1])$. $F_{L,k} : E_k \to E_{k+1}$ is defined by

$$F_{L,k}(g) = \phi_{k+1} G_{L,k} \phi_k^{-1}(g) \qquad \text{for } g \in E_k.$$

Let $\prod E_k$ be the product space of $\{E_k\}$ and $E_\infty = \{\{g_k\} \in \prod E_k | \exists \lim_{k\to\infty} g_k \text{ in } USC(\boldsymbol{R}^d \times [0,1])\}$. The following relation holds:

$$\begin{array}{ccc} E_k & \xrightarrow{F_{L,k}} & E_{k+1} \\ \phi_k \big\uparrow & & \big\uparrow \phi_{k+1} \\ \mathcal{P}^{d+1}_{+,2^k} & \xrightarrow[G_{L,k}]{} & \mathcal{P}^{d+1}_{+,2^{k+1}} \end{array}$$

$\bar{F}_L : \prod E_k \to \prod E_k$ is defined by

$$\bar{F}_L(\bar{g}) = \{\lambda_k\}_{k=0}^{\infty} \text{ for } \bar{g} = \{g_k\},$$

where

$$\begin{cases} \lambda_0 & = & g_0 \\ \lambda_{k+1} & = & F_{L,k}(g_k) \quad k \geq 0 \end{cases}.$$

The distance $d(\bar{g}, \bar{h})$ between $\bar{g} = \{g_k\}$ and $\bar{h} = \{h_k\} \in \prod E_k$ is defined by

$$d(\bar{g}, \bar{h}) = \sum_{k=0}^{\infty} \frac{1}{2^k} d(g_k, h_k),$$

where

$$d(g_k, h_k) = \begin{cases} 1 & g_k \neq h_k \\ 0 & g_k = h_k \end{cases}.$$

For $\{\bar{g}^n\}_n \subset \prod E_k$ with $\bar{g}^n = \{g^n_k\}_k$, we shall define $\lim_{n\to\infty} \bar{g}^n$ in $\prod E_k$ by $\bar{h} \in \prod E_k$ if $\lim_{n\to\infty} d(\bar{g}^n, \bar{h}) = 0$. The following theorem holds.

Theorem 1. *The following statements hold :*

(a) $\bar{F}_L$ *is a contraction on the metric space* $H_a := \{\bar{g} = \{g_k\} \in \prod E_k \mid g_0 = \phi_0(a)\}$ *for any finite and nonzero* $a \in \mathcal{P}^d$.

(b) *There exists* $\lim_{n\to\infty} \bar{F}_L^n \bar{g}$ *in* $\prod E_k$ *for any* $\bar{g} \in \prod E_k$.

(c) $\lim_{n\to\infty} \bar{F}_L^n \bar{g}$ *in* $\prod E_k$ *is* $\bar{F}_L-$ *invariant.*

(d) *The following* (d-1) *and* (d-2) *are equivalent for finite and nonzero* $a \in \mathcal{P}^d$:

(d-1) *There exists* $\lim_{n\to\infty} G_L^k a$ *in* $USC(\boldsymbol{R}^d \times [0,1])$.

(d-2) $\lim_{n\to\infty} \bar{F}_L^n \bar{g}$ *in* $\prod E_k$ *belongs to* E_∞ *for* $\bar{g} = \{\phi_0(a), 0, 0, \cdots\}$.

Since there isn't a one-to-one correspondence between the set $\{\lim_{n\to\infty} G_L^n a \mid a \in \mathcal{P}^d\}$ and the set $\{\bar{h} \in E_\infty \mid \bar{g} = \{\phi_0(a), 0, 0, \cdots\}$ such that $\bar{h} = \lim_{n\to\infty} \bar{F}_L^n \bar{g}\}$, we consider a quotient space. We define the following equivalence relation. The equivalence relation "$\sim$" for $\bar{g} = \{g_k\}$, $\bar{h} = \{h_k\} \in \prod E_k$ is defined by

$$\bar{g} \sim \bar{h} \iff \begin{cases} \lim\limits_{k\to\infty} g_k = \lim\limits_{k\to\infty} h_k & \text{in} \quad USC(\boldsymbol{R}^d \times [0,1]), \\ \text{or} & \\ g_k = h_k & \text{for all} \quad k \in \{0,1,2,\cdots\}. \end{cases} \tag{2.2}$$

Let $\tilde{E} = \prod E_k / \sim$ be a quotient space, $\pi : \prod E_k \to \tilde{E}$ be the canonical quotient map. Because $\bar{g} \sim \bar{h}$ implies $\bar{F}_L \bar{g} \sim \bar{F}_L \bar{h}$, we can define a map $\tilde{F}_L : \tilde{E} \to \tilde{E}$ by

$$\tilde{F}_L(\pi \bar{g}) = \pi(\bar{F}_L \bar{g}).$$

$$\begin{array}{ccc} \prod E_k & \xrightarrow{\bar{F}_L} & \prod E_k \\ \pi \downarrow & & \downarrow \pi \\ \tilde{E} & \xrightarrow[\tilde{F}_L]{} & \tilde{E} \end{array}$$

3. Linear rules

In this section we show that there exists $\lim_{n\to\infty} G_L^n a$ in $USC(\boldsymbol{R}^1 \times [0,1])$ for L and the limit set is independent of an initial configuration a.

Theorem 2. *Let* L *be a linear modulo 2 and* $d = 1$. *Then for a finite nonzero configuration* $a \in \mathcal{P}^1$ *there exists a limit set* $\lim_{n\to\infty} G_L^n a$ *in* $USC(\boldsymbol{R}^1 \times [0,1])$, *which is independent of* a.

Theorem 2 follows from Lemma 1.

Lemma 1. *Suppose a map* $G_L^k : \mathcal{P}^1 \to USC(\boldsymbol{R}^1 \times [0,1])$ *is defined by the equation* (2.1) *and* $a \in \mathcal{P}^1$ *is finite and nonzero. Then the following are true* :

(a) $\displaystyle\bigwedge_{n=1}^{\infty} \bigvee_{k\geq n} G_L^k(\delta_0) = \bigwedge_{n=1}^{\infty} \bigvee_{k\geq n} G_L^k(a)$;

(b) $\displaystyle\bigvee_{n=1}^{\infty} \bigwedge_{k\geq n} G_L^k(\delta_0) = \bigvee_{n=1}^{\infty} \bigwedge_{k\geq n} G_L^k(a)$;

(c) $\displaystyle\bigwedge_{n=1}^{\infty} \bigvee_{k\geq n} G_L^k(\delta_0) = \bigvee_{n=1}^{\infty} \bigwedge_{k\geq n} G_L^k(\delta_0)$.

The next theorem follows from Theorem 1(c) and Theorem 2.

Theorem 3. *Let* L *be linear modulo 2 and* $d = 1$ *and* $\bar{g} = \{g_k\} \in \prod E_k$. *If* $g_0 \in USC(\boldsymbol{R}^1 \times [0,1])$ *has a compact support and nonzero, then there exists* $\lim_{n\to\infty} \bar{F}_L^n \bar{g}$ *in* $\prod E_k$ *and it belongs to* E_∞.

The next theorem follows from Theorem 2 and Theorem 3.

Theorem 4. *Let* L *be linear modulo 2 and* $d = 1$.

(a) *The $\tilde{F}_L$-invariant set in $\pi(E_\infty)$ consists of one element $\tilde{h}$.*

(b) *For any $\bar{g} = \{g_k\} \in \prod E_k$, there exists the limit set $\lim_{n\to\infty} \tilde{F}_L^n(\pi\bar{g})$, which is equal to $\tilde{h}$ in* (a), *if $g_0 \neq 0$ has a compact support.*

4. Non-linear rules

Consider a transition rule $L:\mathcal{P}^1 \to \mathcal{P}^1$ mod 2 as follows:

$$\begin{aligned} La(x) &= \sum_{k=1}^{m} \alpha_k a(x+v_k) + \sum_{i<j} \beta_{i,j} a(x+v_i)a(x+v_j) \\ &\quad + \sum_{i_1<i_2<i_3} \gamma_{i_1,i_2,i_3} a(x+v_{i_1})a(x+v_{i_2})a(x+v_{i_3}) \\ &= L_0 a(x) + L_1 a(x) + L_2 a(x), \end{aligned}$$

that is, L_0 is linear and L_1 and L_2 are non-linear. Let $A = \{i \,|\, \alpha_i \neq 0\}$, $B = \{(i,j) \,|\, \beta_{i,j} \neq 0\}$, $C = \{(i_1,i_2,i_2) \,|\, \gamma_{i_1,i_2,i_3} \neq 0\}$, then we can rewrite

$$\begin{aligned} La(x) &= \sum_{k\in A} a(x+v_k) + \sum_{(i,j)\in B} a(x+v_i)a(x+v_j) \\ &\quad + \sum_{(i_1,i_2,i_3)\in C} a(x+v_{i_1})a(x+v_{i_2})a(x+v_{i_3}). \end{aligned} \tag{4.1}$$

Let $\bar{\delta}_0 = \{\phi_0(\delta_0), 0, 0, \cdots\} \in \prod E_k$. We shall investigate conditions of L and an initial configuration g_0 such that $\lim_{n\to\infty} \bar{F_L}^n \bar{g}$ in $\prod E_k$ belongs to E_∞ for $\bar{g} = \{\phi_0(g_0), 0, 0, \cdots\}$.

We define

$$V_q = \left\{ \sum_{j=0}^{k} \delta_{n_j\cdot q} | k \geq 0, n_0 = 0, n_j \geqq 1, (j \geq 1), n_{j+1} > n_j \right\},$$

$m(b) = n_k \cdot q$ for $b = \sum_{j=0}^{k} \delta_{n_j \cdot q} \in V_q$ and

$$W = \left\{ \sum_{j=1}^{k} \delta_j | k \geq 1 \right\}.$$

Proposition 1. *The following statements hold.*

(1) *If L is linear, for $n \in \boldsymbol{N}, a \in \mathcal{P}^1$*

$$L^n a(x) = \sum_{s_1\in A} \sum_{s_2 \in A} \cdots \sum_{s_n\in A} a(x + v_{s_1} + \cdots + v_{s_n}).$$

(2) *Suppose $C = \emptyset$ for C in* (4.1). *If there is $q \geqq 2$ such that*

(i) *$k_1, k_2 \in A$ implies $q|(v_{k_1} - v_{k_2})$,*

(ii) *$(i,j) \in B$ implies $0 < |v_i - v_j| < q$,*

then

$$L^n g_0(x) = L_0^n g_0(x) \textit{ holds for any } x \in \boldsymbol{Z}, n \in \boldsymbol{N}, g_0 \in V_q.$$

Theorem 5 follows from Proposition 1 and Theorem 3.

Theorem 5. *Suppose a transition rule L is defined by* (4.1) *and satisfies the following properties: There is $q \geqq 2$ such that*

(i) $k_1, k_2 \in A$ *implies* $q|(v_{k_1} - v_{k_2})$;

(ii) $(i,j) \in B$ *implies* $0 < |v_i - v_j| < q$;

(iii) $C = \emptyset$.

Then there exists $\tilde{h} \in \pi(E_\infty)$ such that $\lim_{n\to\infty} \tilde{F}_L^n(\pi\bar{g}) = \tilde{h}$ holds for any $\bar{g} = \{\phi_0(g_0), 0, 0, \cdots\}$ with $g_0 \in V_q$.

Lemma 2. *Let L satisfy the following properties:*

(a) *There is $q \geqq 2$ such that*

$$q|(v_{j_{l+1}} - v_{j_l}) \quad (1 \le l \le M-1),$$

where $A = \{j_1, \cdots, j_M\}$ $(j_1 < \cdots < j_M)$;

(b) $B = \{(i,j) \,|\, v_i = v_j - 1$ *for* $j \in \{j_2, \cdots, j_M\}\}$;

(c) $C = \emptyset$.

If $c \in W$, then $Lc = L_0\delta_1 + \sum_{t=2}^{m(c)} \delta_{-v_{j_1}+t}$ holds.

Proposition 2. *Let L satisfy the same conditions as in Lemma 2. If $g_0 = b + (c\tilde{+}m(b))$ for $c \in W$, $b \in V_q$, then*

$$L^n g_0 = L_0^n b + (c\tilde{+}(m(b) - nv_{j_1})). \tag{4.2}$$

We define the set J by

$$J = \{g_0 \mid g_0 = b + (c\tilde{+}m(b)) \quad \text{for} \quad c \in W,\ b \in V_q\}.$$

Theorem 6 follows from Proposition 2 and Theorem 3.

Theorem 6. *Suppose a transition rule L is defined by* (4.1) *and satisfies the following properties:*

(a) *There is $q \geqq 2$ such that*

$$q|(v_{j_{l+1}} - v_{j_l}) \quad \text{for } 1 \le l \le M-1,$$

where $A = \{j_1, \cdots, j_M\}$ $(j_1 < \cdots < j_M)$;

(b) $B = \{(i,j) \,|\, v_i = v_j - 1$ *for* $j \in \{j_2, \cdots, j_M\}\}$;

(c) $C = \emptyset$.

Then there exists $\tilde{h} \in \pi(E_\infty)$ such that $\lim_{n\to\infty} \tilde{F}_L^n(\pi\bar{g}) = \tilde{h}$ holds for any $\bar{g} = \{\phi_0(g_0), 0, 0, \cdots\}$ with $g_0 \in J$.

We investigate the most simplest non-linear rule which contains the triadic term. We consider the conditions for $\bar{g}$ and L such that $\lim_{n\to\infty} \tilde{F}_L^n(\pi\bar{g})$ exists and it belongs to $\tilde{h} \in \pi(E_\infty)$, when the rule satisfies $v_1 = -r$, $v_2 = -r+1$, $\cdots$, $v_{2r+1} = r$.

Lemma 3. *Let $a \in \mathcal{P}^1$ be finite and nonzero. Suppose a transition rule L is defined by* (4.1) *and satisfies the following properties:*

(a) $A = \{1, 2r+1\}$;

(b) $B = \{(1, r+1)\}$ *or* $B = \{(r+1, 2r+1)\}$;

(c) $C = \{(1, r+1, 2r+1)\}$.

Then

(i) *If* $a(x)a(x+r+2rl)=0$ *for any* $l \in \boldsymbol{N} \cup \{0\}$ *and any* $x \in \boldsymbol{Z}$*, then*

$$L^n a(x) L^n a(x+r+2rl)=0 \qquad (n \in \boldsymbol{N},\ l \in \boldsymbol{N} \cup \{0\}).$$

(ii) *If there is* $M \in \boldsymbol{N}$ *such that*

$$a(x)a(x+r+2rl)=0 \qquad (x \in \boldsymbol{Z},\ 0 \leqq l \leqq M),$$

then

$$L^k a(x) L^k a(x+r+2rl)=0 \quad (k \leq M,\ 0 \leqq l \leqq M-k).$$

Proposition 3. *Suppose a transition rule* L *is defined by* (4.1) *and satisfies the same conditions as in Lemma* 3. *Let* $a \in \mathcal{P}^1$ *be finite and nonzero. The following are equivalent:*

(i) $a(x)a(x+r+2rl)=0$ *holds for any* $l \in \boldsymbol{N} \cup \{0\}$*, any* $x \in \boldsymbol{Z}$*.*

(ii) $L^n a = L_0^n a$ *holds for any* $n \in \boldsymbol{N}$*.*

Thorem 7 follows from Proposition 3 and Theorem 3.

Theorem 7. *Suppose a transition rule* L *is defined by* (4.1) *and satisfies the following properties:*

(a) $A=\{1,2r+1\}$;

(b) $B=\{(1,r+1)\}$ *or* $B=\{(r+1,2r+1)\}$;

(c) $C=\{(1,r+1,2r+1)\}$.

Let $g_0 \in \mathcal{P}^1$ *be finite and nonzero. If* $g_0(x)g_0(x+r+2rl)=0$ *for any* $l \in \boldsymbol{N} \cup \{0\}$ *and any* $x \in \boldsymbol{Z}$*, then there exists* $\tilde{h} \in \pi(E_\infty)$ *such that* $\lim_{n\to\infty} \tilde{F}_L^n(\pi \bar{g}) = \tilde{h}$*.*

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Perfectness of Conelike $*$-Semigroups in $\mathbb{Q}^k$

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Abstract. Let S be an abelian $*$-semigroup in $\mathbb{Q}^k$. We give a sufficient condition for every positive definite function on S to have a unique representing measure on the dual semigroup of S (i.e. S is perfect). To characterize perfectness for any abelian $*$-semigroup is a challenging, but not yet generally solved problem. In this paper, we characterize the structure of involutions on an abelian $*$-semigroup which is a subset of $\mathbb{Q}^k$, and show that any conelike $*$-semigroups in $\mathbb{Q}^k$ are perfect.

1. Preliminaries

Let $S = (S, +, *)$ denote an abelian $*$-semigroup with the identity 0, i.e. S is an abelian semigroup with the identity 0, and equipped with an involution $*$, being a map from S to S such that $(s+t)^* = s^* + t^*$ and $(s^*)^* = s$ for $s, t \in S$. A complex-valued function ρ on S is called a semicharacter if it satisfies

$$\begin{aligned} &\text{(i)} \quad \rho(0) = 1, \\ &\text{(ii)} \quad \rho(s+t) = \rho(s)\rho(t) \quad \text{for } s, t \in S, \\ &\text{(iii)} \quad \rho(s^*) = \overline{\rho(s)} \quad \text{for } s \in S. \end{aligned}$$

The set of all semicharacters is denoted by S^*. We equip S^* with the topology of pointwise convergence. Then S^* is a topological semigroup under pointwise multiplication with involution $\rho \mapsto \overline{\rho}$ and the identity $\mathbf{1}$. S^* is called the dual semigroup of S. Since $\mathbb{C}^S$ is completely regular and S^* is a closed subset of $\mathbb{C}^S$, S^* is a completely regular space. A complex-valued function φ on S is called positive definite if for any $s_1, s_2, \cdots, s_n \in S$ and $c_1, c_2, \cdots, c_n \in \mathbb{C}$

$$\sum_{i,j=1}^{n} c_i \overline{c_j} \varphi(s_i + s_j^*) \geqq 0.$$

The set of all regular Borel measures on S^* is denoted by $M_+(S^*)$. Let $E_+(S^*)$ denote the set of measures $\mu \in M_+(S^*)$ such that

$$\int_{S^*} |\rho(s)| d\mu(\rho) < \infty \quad \text{for } s \in S.$$

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A complex-valued function φ on S is called a *moment* function if there exists a measure $\mu \in E_+(S^*)$ such that

$$\varphi(s) = \int_{S^*} \rho(s) d\mu(\rho) \quad \text{for } s \in S,$$

and μ is called a representing measure for φ. Every moment function is positive definite, and every bounded positive definite function on S is a moment function whose representing measure is unique (see [5, Theorem 2.1]). But a positive definite function is not necessarily a moment function (see [1, Theorem 4]), and a representing measure is not necessarily unique if any (cf. [4], [8]). So an abelian $*$-semigroup S is called perfect if every positive definite function on S is a moment function with a unique representing measure on S^*.

Example. The following are examples of perfect $*$-semigroups:
(i) Abelian $*$-semigroups with finite elements, in particular, the trivial semigroup $\{0\}$.
(ii) Abelian groups with the group involution $s^* = -s$ (by Bochner's Theorem).
(iii) The abelian semigroup $\mathbb{Q}_+$ of nonnegative rational numbers with the identical involution (see [2], Proposition 6.5.6).
(iv) The abelian semigroup $\mathbb{Q}$ with the identical involution (see [2], Proposition 6.5.10).

Perfect $*$-semigroups have some useful properties:

(1) Let $\{S_n\}_{n \geqq 1}$ be a countable family of abelian $*$-semigroups. Then the direct sum

$$\bigoplus_{n=1}^{\infty} S_n := \{(s_1, s_2, \cdots) \mid s_n \in S_n, s_n = 0 \text{ eventually}\}$$

is perfect if and only if every S_n is perfect (see [2], Note VI).

(2) Any $*$-homomorphic image of a perfect $*$-semigroup is perfect (see [2], Theorem 6.5.5).

(3) A $*$-subsemigroup T of an abelian $*$-semigroup S is said to have the ideal property if

$$t + S := \{t + s \mid s \in S\} \subset T \qquad \text{for all} \quad t \in T \setminus \{0\}.$$

Any $*$-subsemigroup with the ideal property of a perfect $*$-semigroup is perfect (see [6], Theorem).

(4) An abelian $*$-semigroup S is called $*$-divisible if every $s \in S$ can be written in the form $s = mt + nt^*$ for some $t \in S$ and nonnegative integers m, n with $m + n \geqq 2$. If a countable abelian $*$-semigroup S is $*$-divisible, then S is perfect (cf. [3], Theorem 4).

Using the properties (1) and (2), we have the following, which is a basic tool for our discussions.

Proposition 1.1. *Let S be an abelian $*$-semigroup with 0, and S_n, $n \geqq 1$, abelian $*$-subsemigroups of S. If every S_n is perfect and $S = \bigcup_{n=1}^{\infty} S_n$, then S is perfect.*

Proof. By the property (1), we have that $\bigoplus_{n=1}^{\infty} S_n$ is perfect. Let $h : \bigoplus_{n=1}^{\infty} S_n \longrightarrow S$ be the $*$-homomorphism defined by

$$h(s_1, s_2, \cdots) := \sum_{n=1}^{\infty} s_n.$$

The assumption $S = \bigcup_{n=1}^{\infty} S_n$ implies that h is onto. By the property (2), we have that S is perfect. $\square$

Definition. Let S be an abelian $*$-semigroup which is a subset of $\mathbb{Q}^k$, $k \geqq 1$. S is called conelike if for every $s \in S$ there is a nonnegative rational number $\alpha(s)$ such that $\alpha s \in S$ for all rational numbers $\alpha \geqq \alpha(s)$ (cf. [7]).

Throughout this paper, the composition on abelian $*$-semigroups in $\mathbb{Q}^k$ is the ordinary addition.

2. Auxiliary Results

In this section, firstly, we consider a conelike subsemigroup of $\mathbb{Q}^k$, $k \geqq 1$, with the identical involution. Next we consider a conelike $*$-subsemigroup of $\mathbb{Q}^2$ with the involution $(p, q)^* = (p, -q)$. The following Proposition 2.1 and Proposition 2.2 are special cases of our main result (Theorem 3.2), and indispensable results to proving Theorem 3.2.

Proposition 2.1. *Let S be a conelike subsemigroup of $\mathbb{Q}^k$, $k \geqq 1$, with the identical involution. Then S is perfect.*

Proof. Firstly let us prove theorem for $k = 1$. If $S \cap (\mathbb{Q}_+ \setminus \{0\}) \neq \emptyset$ and $S \cap (-\mathbb{Q}_+ \setminus \{0\}) \neq \emptyset$, then $S = \mathbb{Q}$. By Example (iv), S is perfect. Then we may assume $S \subset \mathbb{Q}_+$. Since S is conelike, there is a number $a \in \mathbb{Q}_+$ such that $[a, \infty) \cap \mathbb{Q}_+ \subset S$. Define the countable set

$$X := (0, a) \cap \mathbb{Q}_+ \cap S$$

and the semigroup

$$S_x := \langle x \rangle \cup (\mathbb{Q}_+ + a) \quad \text{for } x \in X,$$

where $\langle x \rangle$ is the semigroup generated by x and $\mathbb{Q}_+ + a = \{r \in \mathbb{Q}_+ \mid r \geqq a\}$. Then every S_x is perfect. In fact, put

$$y := \min\{a - nx > 0 \mid n \in \mathbb{N}\}$$

and $y = a - n_0 x$ for some $n_0 \in \mathbb{N}$, then $(\mathbb{Q}_+ + y) \cup \{0\}$ is perfect by Example (iii) and the property (3). Since the subsemigroup $(\mathbb{Q}_+ + (y + x)) \cup \{0, x\}$ of $(\mathbb{Q}_+ + y) \cup \{0\}$ has the ideal property, $(\mathbb{Q}_+ + (y + x)) \cup \{0, x\}$ is perfect. Then we see by iteration that

$$S_x = (\mathbb{Q}_+ + (y + n_0 x)) \cup \{0, x, 2x, \cdots, n_0 x\}$$

is perfect. In addition, we note that $S = \bigcup_{x \in X} S_x$ and $\{S_x\}_{x \in X}$ is a countable family. Hence S is perfect by Proposition 1.1.

Next let us prove theorem for $k \geqq 2$. Define the relation $\equiv$ on $S \setminus \{0\}$ as follows:

$$s \equiv t :\Longleftrightarrow \text{there exists } r \in \mathbb{Q} \setminus \{0\} \text{ such that } s = rt.$$

Then clearly $\equiv$ is an equivalence relation. Let $\{S_\alpha\}_\alpha$ be a family of equivalence classes of $S \setminus \{0\}$ under $\equiv$. For every α, $T_\alpha := S_\alpha \cup \{0\}$ is a conelike semigroup. Since T_α can be identified with a conelike subsemigroup of $\mathbb{Q}$, it follows that T_α is perfect. Furthermore, $\{T_\alpha\}_\alpha$ is a countable family and $S = \bigcup_\alpha T_\alpha$. Therefore S is perfect by Proposition 1.1. □

Proposition 2.2. *Let S be a conelike $*$-subsemigroup of $(\mathbb{Q}^2, +, (p,q)^* = (p,-q))$. Then S is perfect.*

Proof. In case that $S \subset \mathbb{Q} \times \{0\}$ (resp. $S \subset \{0\} \times \mathbb{Q}$), S is perfect by Proposition 2.1 (resp. Example (ii)). In case that S is not contained in any half-plane of $\mathbb{R}^2$, the conelikeness of S implies that $S = \mathbb{Q}^2$. Since $(\mathbb{Q}^k, +, *)$, $k \geqq 1$, with general involution is perfect by the property (4), S is perfect. Accordingly we may assume $S \subset \mathbb{Q}_+ \times \mathbb{Q}$, furthermore, we may assume $S \cap (\{0\} \times \mathbb{Q}) = \{(0,0)\}$. Because if $S \cap (\{0\} \times \mathbb{Q}) \neq \{(0,0)\}$, then the conelikeness of S implies that $S \cap (\{0\} \times \mathbb{Q}) = \{0\} \times \mathbb{Q}$. By Example (ii), $\{0\} \times \mathbb{Q}$ is perfect. We note that $\{0\} \times \mathbb{Q}$ is a face of S, i.e.

$$s + t \in \{0\} \times \mathbb{Q} \text{ and } s, t \in S \text{ imply that } s, t \in \{0\} \times \mathbb{Q}.$$

Since S is perfect if and only if $\{0\} \times \mathbb{Q}$ and $(S \setminus (\{0\} \times \mathbb{Q})) \cup \{(0,0)\}$ are perfect by [6, Theorem 2.1], it suffices to prove that S is perfect when $S \subset \mathbb{Q}_+ \times \mathbb{Q}$ and $S \cap (\{0\} \times \mathbb{Q}) = \{(0,0)\}$. For every $r \in \mathbb{Q}_+ \setminus \{0\}$,

$$S_r := \{(p,q) \in S \mid -rp \leqq q \leqq rp\}$$

is a conelike $*$-subsemigroup of S and $S = \bigcup_{r \in \mathbb{Q}_+ \setminus \{0\}} S_r$. By Proposition 1.1, it suffices to prove that every S_r is perfect. Furthermore, since we take only r such that $\{(p,q) \in S \mid q = rp\} \neq \{(0,0)\}$, we may assume that $\{(p,q) \in S \mid q = rp\} \neq \{(0,0)\}$. To finish the proof we may make use of the following proposition.

Proposition 2.3. *Let $r \in \mathbb{Q}_+ \setminus \{0\}$, and define two abelian $*$-subsemigroups S, T of $(\mathbb{Q}^2, +, (p,q)^* = (p,-q))$ by*

$$\begin{aligned} S &:= \{(p,q) \in \mathbb{Q}_+ \times \mathbb{Q} \mid -rp \leqq q \leqq rp\}, \\ T &:= (S + (1,0)) \cup \{(0,0)\}. \end{aligned}$$

If W is an abelian $*$-subsemigroup of S containing T, then W is perfect.

Proof. W^* is topological semigroup isomorphic to T^*. In fact, $T^* \setminus \{\mathbf{1}_{\{0\}}\}$ is topological semigroup isomorphic to $\{\rho \in S^* \mid \rho_{|T} \neq \mathbf{1}_{\{0\}}\}$ (see [6], p.214). Moreover $\rho \in S^*$ and $\rho_{|T} = \mathbf{1}_{\{0\}}$ is only $\mathbf{1}_{\{0\}} \in S^*$. Because, since for every $s \in S \setminus T$ there exists $n \in \mathbb{N}$ such that $n(s + s^*) \in T$,

$$\begin{aligned} 0 &= \rho(n(s+s^*)) \\ &= \rho(s+s^*)^n \\ &= |\rho(s)|^{2n}, \end{aligned}$$

i.e. $\rho(s) = 0$ for every $s \in S \setminus T$. Therefore $T^* \setminus \{\mathbf{1}_{\{0\}}\}$ is topological semigroup isomorphic to $S^* \setminus \{\mathbf{1}_{\{0\}}\}$. Since $T \subset W \subset S$ and $T^* \setminus \{\mathbf{1}_{\{0\}}\} \cong S^* \setminus \{\mathbf{1}_{\{0\}}\}$, $W^* \setminus \{\mathbf{1}_{\{0\}}\}$ is topological semigroup isomorphic to $T^* \setminus \{\mathbf{1}_{\{0\}}\}$. Corresponded $\mathbf{1}_{\{0\}}$ in W^* to $\mathbf{1}_{\{0\}}$ in T^*, we obtain that W^* is topological semigroup isomorphic to T^*. Let f denote the topological semigroup isomorphism from W^* to T^* such that $f(\sigma) = \sigma_{|T}$, $\sigma \in W^*$.

Let φ be a positive definite function on W with $\varphi(0) = 1$. Since $r + r^* + T \subset W$ for each $r \in W \setminus \{0\}$, $\varphi(r + r^* + \cdot)$ becomes a positive definite function on T. Since S is perfect by the property (4), T is perfect by (3). Then there exists a unique measure $\mu_r \in E_+(T^*)$ such that

$$\varphi(r+r^*+t) = \int_{T^*} \rho(t) d\mu_r(\rho), \quad t \in T. \tag{2.1}$$

Let $\widetilde{\mu}_r \in M_+(W^*)$ be the image measure defined by $\widetilde{\mu}_r := {\mu_r}^{f^{-1}}$. Since $\varphi(0) = 1$, positive definiteness of φ implies that the kernel function

$$(t_1, t_2) \mapsto \varphi(r + r^* + t_1 + t_2^*) - \varphi(r + t_1)\overline{\varphi(r + t_2)}$$

is positive definite on $T \times T$. In other words, for any $t_1, t_2, \cdots, t_n \in T$ and $c_1, c_2, \cdots, c_n \in \mathbb{C}$, it holds that

$$\begin{aligned}
|\sum_{i=1}^{n} c_i \varphi(r+t_i)|^2 &\leqq \sum_{i,j=1}^{n} c_i \overline{c_j} \varphi(r + r^* + t_i + t_j^*) \\
&= \int_{T^*} |\sum_{i=1}^{n} c_i \rho(t_i)|^2 d\mu_r(\rho) \\
&= \int_{W^*} |\sum_{i=1}^{n} c_i \sigma(t_i)|^2 d\widetilde{\mu}_r(\sigma).
\end{aligned}$$

Define, for each $t \in T$, the function $\chi_t : \sigma \mapsto \sigma(t)$ on W^*. Then the mapping $\chi_t \mapsto \varphi(r+t)$, for all $t \in T$, can be extended to a bounded linear functional on $L^2(W^*, \widetilde{\mu}_r)$ with norm $\leqq 1$. Hence there exists an $\widetilde{h}_r \in L^2(W^*, \widetilde{\mu}_r)$ such that $\int_{W^*} |\widetilde{h}_r(\sigma)|^2 d\widetilde{\mu}_r(\sigma) \leqq 1$ and

$$\varphi(r+t) = \int_{W^*} \sigma(t) \overline{\widetilde{h}_r(\sigma)} d\widetilde{\mu}_r(\sigma), \quad t \in T.$$

In particular,

$$\varphi(r + r^* + t) = \int_{W^*} \sigma(t) \overline{\sigma(r) \widetilde{h}_r(\sigma)} d\widetilde{\mu}_r(\sigma), \quad t \in T.$$

Let h_r be the function in $L^2(T^*, \mu_r)$ defined by $h_r(f(\sigma)) := \widetilde{h}_r(\sigma)$, $\sigma \in W^*$. Then

$$\varphi(r + r^* + t) = \int_{T^*} \rho(t) \overline{f^{-1}(\rho)(r) h_r(\rho)} d\mu_r(\rho), \quad t \in T. \tag{2.2}$$

By (2.1) and (2.2), we obtain two signed representing measures μ_r and $\overline{f^{-1}(\rho)(r)h_r(\rho)}\mu_r$ of the moment function $\varphi(r+r^*+t)$. Since for every $r \in W \setminus \{0\}$ there exists $n \in \mathbb{N}$ such that $2^n(r+r^*) \in T$ and $\mu_r(T^*) < \infty$, we have $\overline{f^{-1}(\rho)(r)h_r(\rho)}\mu_r \in E(T^*)$. Then, by perfectness of T, these two signed measures must coincide with each other (see [2, Proposition 6.5.2]). This implies that $\overline{\sigma(r)\widetilde{h}_r(\sigma)}\widetilde{\mu}_r = \widetilde{\mu}_r$ on W^*, so

$$\sigma(r)\widetilde{h}_r(\sigma) = 1 \quad \widetilde{\mu}_r\text{-a.e. on } W^*. \tag{2.3}$$

Hence the measure $\widetilde{\mu}_r$ is concentrated on $G_r := \{\sigma \in W^* \mid \sigma(r) \neq 0\}$, which is an open subset of W^*. Note that $\cup_{r \in W \setminus \{0\}} G_r = W^* \setminus \{\mathbf{1}_{\{0\}}\}$.

Let $r, r' \in W \setminus \{0\}$. Since both $|f^{-1}(\rho)(r')|^2 \mu_r$, $|f^{-1}(\rho)(r)|^2 \mu_{r'} \in E(T^*)$ are representing measures of $\varphi(r + r' + r^* + r'^* + \cdot)$, we obtain, again by perfectness of T, that $|f^{-1}(\rho)(r')|^2 \mu_r = |f^{-1}(\rho)(r)|^2 \mu_{r'}$, i.e. $|\sigma(r')|^2 \widetilde{\mu}_r = |\sigma(r)|^2 \widetilde{\mu}_{r'}$ on W^*. Hence, by (2.3), the measures $|\widetilde{h}_r(\sigma)|^2 \widetilde{\mu}_r$ and $|\widetilde{h}_{r'}(\sigma)|^2 \widetilde{\mu}_{r'}$ coincide with each other on $G_{r+r'} = G_r \cap G_{r'}$. Therefore we can define a measure $\widetilde{\mu}$ on W^* by

$$\widetilde{\mu} := \begin{cases} |\widetilde{h}_r(\sigma)|^2 \widetilde{\mu}_r & \text{on} \quad G_r, \quad r \in W \setminus \{0\}, \\ 0 & \text{off} \quad \cup_{r \in W \setminus \{0\}} G_r, \end{cases}$$

(cf. [2, Theorem 2.1.18]). Obviously, $\widetilde{\mu}$ is concentrated on $W^* \setminus \{\mathbf{1}_{\{0\}}\}$ and satisfies

$$\int_{W^*} |\sigma(r)| d\widetilde{\mu}(\sigma) < \infty, \quad r \in W \setminus \{0\},$$

$$\varphi(r) = \int_{W^*} \sigma(r) d\widetilde{\mu}(\sigma), \quad r \in W \setminus \{0\}. \tag{2.4}$$

Next we show that $\widetilde{\mu}(W^*) \leqq 1$. Since φ is a positive definite function on W and $\varphi(0) = 1$, the kernel function

$$(w_1, w_2) \mapsto \varphi(w_1 + w_2^*) - \varphi(w_1)\overline{\varphi(w_2)}$$

is positive definite on $W \times W$, i.e. for any $w_1, w_2, \cdots, w_n \in W \setminus \{0\}$ and $c_1, c_2, \cdots, c_n \in \mathbb{C}$

$$\begin{aligned} |\sum_{i=1}^{n} c_i \varphi(w_i)|^2 &\leqq \sum_{i,j=1}^{n} c_i \overline{c_j} \varphi(w_i + w_j^*) \\ &= \int_{W^*} |\sum_{i=1}^{n} c_i \sigma(w_i)|^2 d\widetilde{\mu}(\sigma). \end{aligned}$$

This means that the mapping $\chi_w \mapsto \varphi(w)$, for all $w \in W \setminus \{0\}$, can be extended to a bounded linear functional on $L^2(W^*, \widetilde{\mu})$ with norm $\leqq 1$. Hence there exists an $\widetilde{h} \in L^2(W^*, \widetilde{\mu})$ such that $\int_{W^*} |\widetilde{h}(\sigma)|^2 d\widetilde{\mu}(\sigma) \leqq 1$ and

$$\varphi(w) = \int_{W^*} \sigma(w) \overline{\widetilde{h}(\sigma)} d\widetilde{\mu}(\sigma), \quad w \in W \setminus \{0\}. \tag{2.5}$$

Let $\mu \in M_+(T^*)$ be the image measure defined by $\mu := \widetilde{\mu}^f$. Then, by (2.4) and (2.5),

$$\begin{aligned} \varphi(t + w) &= \int_{W^*} \sigma(t)\sigma(w) d\widetilde{\mu}(\sigma) \\ &= \int_{T^*} \rho(t) f^{-1}(\rho)(w) d\mu(\rho), \\ \varphi(t + w) &= \int_{W^*} \sigma(t)\sigma(w)\overline{\widetilde{h}(\sigma)} d\widetilde{\mu}(\sigma) \\ &= \int_{T^*} \rho(t) f^{-1}(\rho)(w) \overline{\widetilde{h}(f^{-1}(\rho))} d\mu(\rho), \end{aligned}$$

for $t \in T, w \in W \setminus \{0\}$. Note that $f^{-1}(\rho)(w)\mu,\ f^{-1}(\rho)(w)\overline{\widetilde{h}(f^{-1}(\rho))}\mu \in E(T^*)$. By perfectness of T, $f^{-1}(\rho)(w)\mu = f^{-1}(\rho)(w)\widetilde{h}(f^{-1}(\rho))\mu$ on T^*, i.e. $\sigma(w)\widetilde{\mu} = \sigma(w)\widetilde{h}(\sigma)\widetilde{\mu}$ on W^*. Then $\widetilde{\mu} = \widetilde{h}(\sigma)\widetilde{\mu}$ on G_w, $w \in W \setminus \{0\}$. Hence $\widetilde{h}(\sigma) = 1$ $\widetilde{\mu}$-a.e. on W^*, and

$$\widetilde{\mu}(W^*) = \int_{W^*} |\widetilde{h}(\sigma)|^2 d\widetilde{\mu}(\sigma) \leqq 1.$$

Put $\widetilde{\nu} := \widetilde{\mu} + (1 - \widetilde{\mu}(W^*))\varepsilon_{\{\mathbf{1}_{\{0\}}\}} \in E_+(W^*)$, where $\varepsilon_{\{\mathbf{1}_{\{0\}}\}}$ denotes the Dirac measure at the point $\mathbf{1}_{\{0\}}$. Then we obtain an integral representation

$$\varphi(r) = \int_{W^*} \sigma(r) d\widetilde{\nu}(\sigma), \quad r \in W.$$

Finally, uniqueness of the representing measure of φ follows from perfectness of T. Hence W is perfect. □

3. Main Theorems

Let S be an abelian $*$-semigroup, which is a subset of $\mathbb{Q}^k$, $k \geqq 1$. The following theorem gives the structure of involutions on S, where $(T, +, E)$ means the abelian $*$-semigroup T with involution $t^* = Et$.

Theorem 3.1. *For every abelian $*$-semigroup $(S, +, *)$ in $\mathbb{Q}^k$, there exist an abelian $*$-semigroup T in some $\mathbb{Q}^m$ and a diagonal matrix $E = diag\{\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_m\}$, where $\varepsilon_i = 1$ or -1, such that $(S, +, *)$ is $*$-isomorphic to $(T, +, E)$.*

Proof. In the following $\mathbb{Q}^k$ will be recognized as a k-dimensional linear space over the scalar field $\mathbb{Q}$. Let Lin$S$ denote the linear subspace spanned by S and take $\{r_1, r_2, \cdots, r_m\}$ as basis of Lin$S$ consisting of elements in S. Then it is easily seen that $\{r_1^*, r_2^*, \cdots, r_m^*\}$ is also a basis of Lin$S$. Let A be the linear transformation on Lin$S$ determined by $r_i^* = Ar_i$, $1 \leqq i \leqq m$. Then we can show that

$$\text{(i) for each } s \in S, s^* = As, \quad \text{(ii) } A^2 = I(\text{identity}).$$

In fact, for each $s \in S$, there exist $n, n_1, n_2, \cdots, n_m \in \mathbb{Z}$ such that

$$ns = n_1 r_1 + n_2 r_2 + \cdots + n_m r_m.$$

As a little manipulation of the equation implies

$$ns^* = n_1 r_1^* + n_2 r_2^* + \cdots + n_m r_m^*,$$

we have

$$\begin{aligned} nAs = Ans &= A(n_1 r_1 + n_2 r_2 + \cdots + n_m r_m) \\ &= n_1 Ar_1 + n_2 Ar_2 + \cdots + n_m Ar_m \\ &= n_1 r_1^* + n_2 r_2^* + \cdots + n_m r_m^* \\ &= ns^*. \end{aligned}$$

Hence (i) holds. (ii) follows from

$$A^2 r_i = A r_i^* = (r_i^*)^* = r_i, \quad 1 \leqq i \leqq m.$$

Now put $P := (I + A)/2$, then (ii) implies $P^2 = P$, that is P is a projection onto $\mathrm{ran}P$ along $\ker P$. Therefore

$$\mathrm{Lin}S = \ker P \dotplus \mathrm{ran}P = \ker P \dotplus \ker(I - P) = \ker(I + A) \dotplus \ker(I - A).$$

If we take a basis of each of subspaces $\ker(I + A)$ and $\ker(I - A)$, the summand is a basis of $\mathrm{Lin}S$ consisting of eigenvectors to eigenvalues 1 or -1. Thus, there exist $t_1, t_2, \cdots, t_m \in \mathrm{Lin}S$ such that

$$At_i = \varepsilon_i t_i, \quad 1 \leqq i \leqq m,$$

where $\varepsilon_i = 1$ or -1. Now define W as the non-singular linear transformation from $\mathrm{Lin}S$ onto $\mathbb{Q}^m$ satisfying

$$e_i = Wt_i, \quad 1 \leqq i \leqq m,$$

where $\{e_1, e_2, \cdots, e_m\}$ means the canonical basis of $\mathbb{Q}^m$, and define $T = WS$, the image of S by W. Then

$$WAW^{-1}e_i = WAt_i = \varepsilon_i Wt_i = \varepsilon_i e_i, \quad 1 \leqq i \leqq m.$$

Let E be the matrix representing WAW^{-1} with respect to the basis $\{e_1, e_2, \cdots, e_m\}$. Then the above equality yields that $E = \mathrm{diag}\{\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_m\}$. To complete the proof it remains to show that $(S, +, *)$ is $*$-isomorphic to $(T, +, E)$ by W. To this end, let $s \in S$ and $t = Ws$. Then

$$Ws^* = WAs = WAW^{-1}t = Et = EWs,$$

which ensures to preserve $*$-structures between S and T. This completes the proof. $\square$

Finally we can prove that any conelike $*$-semigroups with general involution in $\mathbb{Q}^k$ are perfect.

Theorem 3.2. *Let S be a conelike $*$-semigroup with general involution, which is a subset of $\mathbb{Q}^k$, $k \geqq 1$. Then S is perfect.*

Proof. By Theorem 3.1, there exist a conelike $*$-semigroup T in some $\mathbb{Q}^m$ and a diagonal matrix $E = \mathrm{diag}\{\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_m\}$, where $\varepsilon_i = 1$ or -1, such that $(S, +, *)$ is $*$-isomorphic to $(T, +, E)$. We note that $(S, +, *)$ is perfect if and only if $(T, +, E)$ is perfect by the property (2). Then we will show that the conelike $*$-semigroup $(T, +, E)$ is perfect by induction on m.

When $m = 1$, the involution on T is identical involution or group involution. By Proposition 2.1 or Example (ii), T is perfect. When $m = 2$, it suffices to prove that $(T, +, \mathrm{diag}\{1, -1\})$ is perfect, which is proved by Proposition 2.2. Now assume that Theorem 3.2 is valid for $1 \leqq m \leqq n - 1$, $n \geqq 3$, and $T = (T, +, E)$ is a conelike $*$-semigroup, which is a subset of $\mathbb{Q}^n$. From $n \geqq 3$ at least two of ε_i's are the same signature. Therefore it suffices to prove that $(T, +, \mathrm{diag}\{\varepsilon_1, \varepsilon_2, \varepsilon_3, \cdots, \varepsilon_n\})$ is perfect when $\varepsilon_1 = \varepsilon_2$. Define the relation $\equiv$ on $T \setminus \{0\}$ as follows:

$$(s_1, s_2, \cdots, s_n) \equiv (t_1, t_2, \cdots, t_n) :\Leftrightarrow \text{ there exists } r \in \mathbb{Q} \setminus \{0\} \text{ such that } (s_1, s_2) = r(t_1, t_2).$$

Then $\equiv$ is an equivalence relation. Let $\{S_\alpha\}_\alpha$ be a family of equivalence classes of $T \setminus \{0\}$ under $\equiv$. For every α, $T_\alpha := S_\alpha \cup \{0\}$ is a conelike $*$-subsemigroup of T. We can identify T_α with the following conelike $*$-semigroup in $\mathbb{Q}^{n-1}$:

$$\begin{cases} \{(s_1, s_3, \cdots, s_n) \in \mathbb{Q}^{n-1} \mid (s_1, s_2, \cdots, s_n) \in T_\alpha\} & \text{if } s_1 \neq 0; \\ \{(s_2, s_3, \cdots, s_n) \in \mathbb{Q}^{n-1} \mid (s_1, s_2, \cdots, s_n) \in T_\alpha\} & \text{if } s_1 = 0. \end{cases}$$

Then every T_α is perfect by the assumption of induction. Furthermore, $\{T_\alpha\}_\alpha$ is a countable family and $T = \bigcup_\alpha T_\alpha$. Therefore T is perfect by Proposition 1.1. $\square$

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A Hlawka type inequality and its converse

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Abstract. A Hlawka type inequality on a Banach space and its converse are introduced and the best constant is given in case of a Hilbert space.

n 次元 Euclid 空間 R^n の任意の元 x, y, z に対して、常に

$$\|x+y\|+\|y+z\|+\|z+x\| \le \|x\|+\|y\|+\|z\|+\|x+y+z\|$$

が成り立つ。これが良く知られた Hlawka 不等式 (cf. [1]) であるが、これは任意の（複素）Hilbert 空間 H に対しても成り立つことが知られている。我々は [4] において、Banach 空間 X 及び指数 $s, r \ge 1$ が与えられたとき、X の任意の元 x, y, z に対して Hlawka 型不等式

$$\left(\|x+y\|^s+\|y+z\|^s+\|z+x\|^s\right)^{1/s} \le C\left(\|x\|^r+\|y\|^r+\|z\|^r+\|x+y+z\|^r\right)^{1/r}$$

が成り立つ最適定数 $C = C(s, r; X)$ を調査し、そのような定数について、

$$C(s,r;X) \le 2^{1-2/r}\cdot 3^{1/s}\ ,\ C(s,r;\ell_n^\infty) = 2^{1-2/r}\cdot 3^{1/s}\ (3 \le n \le \infty)$$

が成り立つこと、また、X が Hilbert 空間であるとき、次が成り立つことを示した。

Theorem 1. Let H be a Hilbert space. Then

(i) $C(s,r;H) = 2^{1-2/r}$ for $2 \le s < \infty, \dfrac{s}{s-1} \le r < \infty$.

(ii) $C(s,r;H) = 2^{1-2/r}\cdot 3^{1/s-1/2}$ for $1 \le s \le 2 \le r < \infty$.

(iii) $C(s,r;H) = 2\cdot 3^{1/s}(3^r+3)^{-1/r}$ for $1 \le s \le r \le 2$.

我々は、この逆問題を考察する。つまり、Banach 空間 X 及び指数 $s, r \ge 1$ が与えられたとき、X の任意の元 x, y, z に対して、

$$C'\left(\|x+y\|^r+\|y+z\|^r+\|z+x\|^r\right)^{1/r} \ge \left(\|x\|^s+\|y\|^s+\|z\|^s+\|x+y+z\|^s\right)^{1/s} \qquad (*)$$

が成り立つ定数 C' は存在するのか？また存在するとしたら、その最適定数は何なのか？と言うことである。

実はそのような定数 C' は常に存在し、$C' \le 2^{-1+2/s}\cdot 3^{1-1/r}$ ととれる。実際任意の $a, b, c \in X$ に対して、

$$\left(\|a-b+c\|^s+\|a+b-c\|^s+\|-a+b+c\|^s+\|a+b+c\|^s\right)^{1/s} \le 4^{1/s}\left(\|a\|+\|b\|+\|c\|\right)$$

$$\le 4^{1/s}3^{1-1/r}\left(\|a\|^r+\|b\|^r+\|c\|^r\right)^{1/r}$$

であるから、変数変換：

$$x=a-b+c,\ y=a+b-c,\ z=-a+b+c$$

によって、上式は

$$\left(\|x\|^s+\|y\|^s+\|z\|^s+\|x+y+z\|^s\right)^{1/s} \le 4^{1/s}3^{1-1/r}\left(\left\|\frac{x+y}{2}\right\|^r+\left\|\frac{y+z}{2}\right\|^r+\left\|\frac{z+x}{2}\right\|^r\right)^{1/r}$$

$$=2^{-1+2/s}3^{1-1/r}\left(\|x+y\|^r+\|y+z\|^r+\|z+x\|^r\right)^{1/r}$$

となるからである。従って最適定数を $C'(r,s;X)$ と書けば、

$$C'(r,s;X)\le 2^{-1+2/s}\cdot 3^{1-1/r}$$

である。更に $C'(r,s;\ell_n^1)=2^{-1+2/s}\cdot 3^{1-1/r}$ $(3\le n\le\infty)$ であることを導く。実際、

$$x=(-1,1,1,0,0,\ldots),\ y=(1,-1,1,0,0,\ldots),\ z=(1,1,-1,0,0,\ldots)$$

は最適定数を実現させるベクトルとなっている。しかしXがHilbert空間であるときはもっとシャープな定数が予想され、実際次の様な定理が成り立つ。

Theorem 2. Let H be a Hilbert space. Then

(i) $C'(r,s;H)=2^{-1+2/s}$ for $1\le r\le 2, s\le \frac{r}{r-1}$.

(ii) $C'(r,s;H)=2^{-1+2/s}\cdot 3^{1/2-1/r}$ for $1\le s\le 2\le r<\infty$.

(iii) $C'(r,s;H)=\dfrac{(3^s+3)^{1/s}}{2}\cdot 3^{-1/r}$ for $2\le s\le r<\infty$.

証明はTheorem 1と同様変数変換をすることによって、type - cotype 理論に持ち込み、そこで、(p, p')-Clarkson type inequality (Kato-Takahashi [3]), Figiel-Iwaniec-Pelczynski [2] の結果等を利用してなされる。

Theorem 1は、指数領域： $\min\left(s,\frac{s}{s-1}\right)\le r$ について最適定数 $C(s,r;H)$ を決定しものであるが、残された領域：$\min\left(s,\frac{s}{s-1}\right)>r$ については困難が予想された。同様にTheorem 2は、指数領域：$s\le\max\left(r,\frac{r}{r-1}\right)$ について最適定数 $C'(r,s;H)$ を決定したものであるが、やはり残された領域：$s>\max\left(r,\frac{r}{r-1}\right)$ については、困難が予想される。

一般にtype不等式とcotype不等式の間に次のような双対定理が成り立つ。

Theorem 3. Let X be a Banach space and X* its dual space. Assume $1<p\le 2, 1<s<\infty, \frac{1}{p}+\frac{1}{p'}=1$ and $\frac{1}{s}+\frac{1}{s'}=1$. If for some constant K, the type inequality :

$$\left(E\left\|\sum_{j=1}^{n}\varepsilon_j x_j\right\|^s\right)^{1/s} \le K\left(\sum_{j=1}^{n}\|x_j\|^p\right)^{1/p} \quad (x_1,\dots,x_n \in X)$$

holds, then for same constant K, the cotype inequality :

$$\left(\sum_{j=1}^{n}\|f_j\|^{p'}\right)^{1/p'} \le K\left(E\left\|\sum_{j=1}^{n}\varepsilon_j f_j\right\|^{s'}\right)^{1/s'} \quad (f_1,\dots,f_n \in X^*)$$

holds.

Hlawka 型不等式は、変数変換によって cotype 不等式に変わるので、上の定理は次の様な双対定理を導く。

Theorem 4. Let X be a Banach space and X* its dual space. Assume $1<p\le 2,\ 1<s<\infty,\ \frac{1}{p}+\frac{1}{p'}=1$ and $\frac{1}{s}+\frac{1}{s'}=1$. If for some constant C, the inequality :

$$C\left(\|x+y\|^r+\|y+z\|^r+\|z+x\|^r\right)^{1/r} \ge \left(\|x\|^s+\|y\|^s+\|z\|^s+\|x+y+z\|^s\right)^{1/s} \quad (x,y,z\in X)$$

holds, then the same constant C, the inequality :

$$\left(\|f+g\|^{r'}+\|g+h\|^{r'}+\|h+f\|^{r'}\right)^{1/r'} \le C\left(\|f\|^{s'}+\|g\|^{s'}+\|h\|^{s'}+\|f+g+h\|^{s'}\right)^{1/s'} \quad (f,g,h\in X^*)$$

holds.

注意：Theorem 2 の (i), (ii) と Theorem 4 から、Theorem 1 の (i), (ii) が導かれる。しかし (iii) は出ない。これは非常に不思議な現象であり、今のところ何故なのか分からない。解明が急がれる。

最後に Banach 空間 X に対する Hlawka 型逆不等式の最適定数 $C'(r,s;X)$ は、作用素論的に考えると、次のような意味を持つことに注意したい：行列

$$T=\begin{bmatrix} 1 & -1 & 1 \\ 1 & 1 & -1 \\ -1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

を Banach 空間 $\ell_3^r(X)$ から Banach 空間 $\ell_4^s(X)$ への線形作用素と考えるとき、これは有界で且つ単射である。このとき変数変換：

$$x=a-b+c,\ y=a+b-c,\ z=-a+b+c$$

を考えると、不等式 (*) は

$$2C'\left\|(a,b,c)\right\|_{\ell_3^r(X)} \le \left\|T(a,b,c)\right\|_{\ell_4^s(X)}$$

となる。このことは、

$$C'(r,s;X)=\frac{1}{2}\|T\|$$

であることを示している。従って最適定数 $C'(r,s;X)$ を求めることは、ある種の具体的な作用素ノルムを求めていることになる。

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Remarks on Williams-Wells' inequality

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In this short note, we shall give some generalizations of the foll-wing L_p-inequality due to Williams and Wells [11]:

Theorem W (Williams and Wells [11])

Let $1 \leqq p \leqq \infty$, $1/p + 1/p' = 1$, $\omega = \exp(2\pi i/n)$ $(i = \sqrt{-1})$.
Then for any t, $1 \leqq t \leqq \min\{p, p'\}$, and for any $x_1, x_2, \ldots, x_n$ in L_p, it holds

$$(\mathrm{WWI}_t) \qquad \left(\sum_{j=1}^{n} \Big\| \sum_{k=1}^{n} \omega^{jk} x_k \Big\|^{t'} \right)^{1/t'} \leqq n^{1/t'} \left(\sum_{k=1}^{n} \| x_k \|^{t} \right)^{1/t}$$

Remark. (1) $n = 2$, $t = \min\{p, p'\}$のときは、(WWI_t)はClarkson不等式である(cf. [2])。したがって、定理WはClarksonの一般化である。

(2) $a_{jk} = \omega^{jk}$とおくと、n次正方行列$A = (a_{jk})$は次の条件を満たす複素行列である。($n = 2$のときは、実行列。)

$$(*) \qquad |a_{jk}| = 1 \ (j, k=1, 2, \ldots, n), \text{ and } \textstyle\sum_j a_{jk} a_{j\ell} = n\delta_{k\ell}.$$

In the following, we assume that $A = (a_{jk})$ is an $n \times n$ (real or complex) matrix satisfying $(*)$. It is easy to see that for any $x_1, x_2, \ldots, x_n$ in L_2, it holds

$$\left(\sum_{j=1}^{n} \Big\| \sum_{k=1}^{n} a_{jk} x_k \Big\|^{2} \right)^{1/2} = n^{1/2} \left(\sum_{k=1}^{n} \| x_k \|^{2} \right)^{1/2}$$

Theorem 1. Let $1 \leqq p \leqq \infty$ and $1 \leqq t \leqq \min\{p, p'\}$. Then for any $x_1, x_2, \ldots\ldots, x_n$ in L_p, it holds

$$(**) \qquad \left(\sum_{j=1}^{n} \| \sum_{k=1}^{n} a_{jk} x_k \|^{t'} \right)^{1/t'} \leqq n^{1/t'} \left(\sum_{k=1}^{n} \| x_k \|^{t} \right)^{1/t}$$

Remark. Theorem 1 はTheoremWの一般化である。

Now we consider Banach spaces X satisfying ($**$). We say that X satisfies (WWI_t), $1 \leqq t \leqq 2$, for $A = (a_{jk})$ if ($**$) holds in X. It is easy to see tat X satisfies (WWI_2) for each (some) A if and only if X is isometric to a Hilbert space, see Jordan and Neumann [5].

Proposition 2. Let $1 \leqq q \leqq p \leqq 2$. If X satisfies ($WWI_p$) for A, then X satisfies (WWI_q) for A.

Remark. これより、(WWI_t)がカラーに対して成立することが分かる。

Theorem 3. Let $1 \leqq p \leqq 2$ and $p \leqq r \leqq p'$. If X satisfies (WWI_p) for A, then any Lebesgue-Bochner space $L_r(X)$ satisfies (WWI_p) for A.

Theorem 4. Let $1 \leqq p \leqq 2$. If X satisfies (WWI_p) for A, then any Lebesgue-Bochner space $L_r(X)$ satisfies (WWI_t) for A, where $1 \leqq t \leqq \min\{p, r, r'\}$.

Corollary 5. $L_q(L_p)$ satisfies (WWI_t) for any A (satisfying ($*$)), where $1 \leqq t \leqq \min\{p, p', q, q'\}$.

Let $A_1 = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$ be a 2×2 Littlewood matrix. Then Clarkson [2] proved that L_p satisfies (WWI_t) for A_1, where $t = \min\{p, p'\}$.
We note that the Clarkson inequality (CI_t) holds in a Banach space X if and only if X satisfies (WWI_t) for A_1.

Corollary 6 (Takahashi and Kato [10]). Let $1 \leqq p \leqq 2$ and suppose that Clarkson inequality (CI_p) holds in X. Then (CI_t) holds in any Lebesgue-Bochner space $L_r(X)$, where $1 \leqq t \leqq \min\{p, r, r'\}$.

Let A_n (n=1, 2, . . .) be $2^n \times 2^n$ Littlewood matrices defined by

$$A_{n+1} = \begin{pmatrix} A_n & A_n \\ A_n & -A_n \end{pmatrix} \quad (n=1, 2, \ldots)$$

where A_1 is a 2×2 Littlewood matrix. We say that the generalized Clarkson inequality (GCI_t), $1 \leqq t \leqq 2$, holds in X if X satisfies (WWI_t) for A_n (n=1, 2, ...). In [6] Kato proved that (GCI_t) holds in L_p, where $1 \leqq t \leqq \min\{p, p'\}$.

Corollary 7 (Hashimoto, Kato and Takahashi [4]). Let $1 \leqq p \leqq 2$ and suppose that generalized Clarkson inequality (GCI_p) holds in X. Then (GCI_t) holds in any Lebesgue-Bochner space $L_r(X)$, where $1 \leqq t \leqq \min\{p, r, r'\}$.

Remarks. Clarkson不等式の一般化は、様々な形でなされている。パラメータを一般化する方向の研究としては、Boas[1], Koskela[9]などがある。この場合、要素の個数は２個のままであるが、個数を2^n個に一般化したものがKato[6]による一般Clarkson不等式(GCI)である。（スカラーの場合に(GCI)はPietschが証明した。）ここで紹介したWilliams-Wellsの不等式は、個数をn個にする一般化ともみなせる。また、Lebesgue-Bochner空間（Sobolev, Bezov, Triebel-Sobolev等を含む）におけるClarkson不等式、あるいはその一般化も様々になされている。最近の研究で、Clarkson不等式とタイプ・コタイプ定数との関連も得られている（Kato and Takahashi [8]）。

我々がここで紹介した結果は、Williams-Wellsの不等式を一般化すると共に(CI), (GCI)等の一般化でもある。（紙面の関係で文献をかなり省略した。）

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Hanner 型不等式への complex weight の導入

An introduction of the complex weight to the Hanner type inequality

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Abstract : In this short note, we prove the n-element version of the Pavlović's inequality as follows:

$$\mathbf{E}\left\|\sum_{i=1}^{n} \varepsilon_i w_i x_i\right\|^p \geq \mathbf{E}\left|\sum_{i=1}^{n} \varepsilon_i w_i \|x_i\|\right|^p \text{ for } 1 \leq p \leq 2, \text{ and}$$

$$\mathbf{E}\left\|\sum_{i=1}^{n} \varepsilon_i w_i x_i\right\|^p \leq \mathbf{E}\left|\sum_{i=1}^{n} \varepsilon_i w_i \|x_i\|\right|^p \text{ for } 2 \leq p < \infty,$$

where n is a natural number, $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n$ are independent Rademacher random variables, $x_1, x_2, \ldots, x_n$ are function in $\mathbf{L}^p_{\mathbf{R}}$ and $w_1, w_2, \ldots, w_n$ are complex number such that $|w_i| = 1$.

The original 2-element Pavlovic's inequality can be regarded as a variation of the Hanner's inequality. We have proved the n-element version of the Hanner's inequality. Introducing the complex weight in the n-element Hanner's inequality, we obtain the n-element version of the Pavlovic's inequality.

重み付き Hanner の不等式 (Pavlović[1996])

$$w \in \mathbf{C} \quad |w| = 1 \quad \forall x_1, x_2 \in L^p_{\mathbf{R}}(S, \Sigma, \mu)$$
$$1 \leq p \leq 2 \Rightarrow \|x_1 + w x_2\|^p + \|x_1 - w x_2\|^p \geq \left|\|x_1\| + w\|x_2\|\right|^p + \left|\|x_1\| - w\|x_2\|\right|^p$$
$$2 \leq p < \infty \Rightarrow \|x_1 + w x_2\|^p + \|x_1 - w x_2\|^p \leq \left|\|x_1\| + w\|x_2\|\right|^p + \left|\|x_1\| - w\|x_2\|\right|^p$$

ε_i を Rademacher 列（各 ε_i は確率 $\frac{1}{2}$ で ± 1 をとる）とすると上式は

$$w_1, w_2 \in \mathbf{C} \quad |w_i| = 1 \quad \forall x_1, x_2 \in L^p_{\mathbf{R}}(S, \Sigma, \mu)$$
$$1 \leq p \leq 2 \Rightarrow \mathbf{E}\left\|\sum_{i=1}^{2} \varepsilon_i w_i x_i\right\|^p \geq \mathbf{E}\left|\sum_{i=1}^{2} \varepsilon_i w_i \|x_i\|\right|^p$$

$$2 \le p < \infty \Rightarrow \mathbf{E}\left\| \sum_{i=1}^{2} \varepsilon_i w_i x_i \right\|^p \le \mathbf{E}\left| \sum_{i=1}^{2} \varepsilon_i w_i \| x_i \| \right|^p$$

となるので、次のように n 要素へ拡張する。

n を自然数，$\varepsilon_1, \varepsilon_2, \dots, \varepsilon_n$ を独立な Rademacher 列，$w_1, w_2, \dots, w_n \in \mathbf{C}$ $|w_i| = 1$, $x_1, x_2, \dots, x_n \in L_{\mathbf{R}}^p$ とする。

$$1 \le p \le 2 \Rightarrow \mathbf{E}\left\| \sum_{i=1}^{n} \varepsilon_i w_i x_i \right\|^p \ge \mathbf{E}\left| \sum_{i=1}^{n} \varepsilon_i w_i \| x_i \| \right|^p$$

$$2 \le p < \infty \Rightarrow \mathbf{E}\left\| \sum_{i=1}^{n} \varepsilon_i w_i x_i \right\|^p \le \mathbf{E}\left| \sum_{i=1}^{n} \varepsilon_i w_i \| x_i \| \right|^p$$

Lemma 1

$1 \le p < \infty$, $\{\varepsilon_i\}_{i=1}^{n}$ Rademacher 列，$u_1, u_2, \dots, u_n \in \mathbf{R}$, $w_1, w_2, \dots, w_n \in \mathbf{C}$, $|w_i| = 1$

$$\Rightarrow \mathbf{E}\left| \sum_{i=1}^{n} \varepsilon_i w_i u_i \right|^p \ge \mathbf{E}\left| \sum_{i=1}^{n} \varepsilon_i w_i | u_i | \right|^p$$

Proof.

$$\begin{aligned} \textit{left-hand side} &= \frac{1}{2^n} \sum_{\substack{\text{全ての}\pm\text{の} \\ \text{とり方}}} \left| \pm w_1 u_1 \pm w_2 u_2 \pm \dots \pm w_n u_n \right| \\ &= \frac{1}{2^n} \sum_{\substack{\text{全ての}\pm\text{の} \\ \text{とり方}}} \left| \pm w_1 | u_1 | \pm w_2 | u_2 | \pm \dots \pm w_n | u_n | \right| \\ &= \textit{right-hand side} \end{aligned}$$

Lemma 2

$u \ge 0, \alpha \in \mathbf{C}, f(u) = \left| u^{\frac{1}{p}} + \alpha \right|^{\frac{p}{2}} + \left| u^{\frac{1}{p}} - \alpha \right|^{\frac{p}{2}}$

このとき

$1 \le p \le 2 \Rightarrow f(u)$ は下に凸

$2 \le p < \infty \Rightarrow f(u)$ は上に凸

Proof.

α が実数の場合は Hanner[1]。ここでは $\alpha = a + ib \in \mathbf{C}$ である。したがって、$f(u) = \left| (u^{\frac{1}{p}} + a)^2 + b^2 \right|^p + \left| (u^{\frac{1}{p}} + a)^2 - b^2 \right|^p$ である。直接 $f''(u) \ge 0$ (≤ 0) を示すことは難しい。複素数 $\mathbf{C}$ を $L_{\mathbf{R}}^p$ に埋め込むことにより α が real の場合に帰着させる。

Lemma 3 (Kigami, Okazaki and Takahashi[2])

g_1, g_2 独立ガウス型確率変数 ～ N(0, 1)

$\varphi : \mathbf{C} \to \boldsymbol{L}_{\mathbf{R}}^{\mathrm{p}}(\Omega, \mathrm{P})$

$\quad z = u + iv \to c_p(ug_1(w) + vg_2(w))$

$$c_p = \left(\int |g_1|^p \, dP(w)\right)^{-\frac{1}{p}}$$

$\Rightarrow$ (i) φ は real linear

(ii) φ は isometry

Lemma 3 の φ により、$f(u)$ は次のように表される。

$$\begin{aligned} f(u) &= |u^{\frac{1}{p}} + \alpha|^p + |u^{\frac{1}{p}} - \alpha|^p \\ &= \|\varphi(u^{\frac{1}{p}} + \alpha)\|^p + \|\varphi(u^{\frac{1}{p}} - \alpha)\|^p \\ &= \|u^{\frac{1}{p}}\varphi(1) + \varphi(\alpha)\|^p + \|u^{\frac{1}{p}}\varphi(1) - \varphi(\alpha)\|^p \\ &= \int \left[|u^{\frac{1}{p}}\varphi(1)(\omega) + \varphi(\alpha)(\omega)|^p + |u^{\frac{1}{p}}\varphi(1)(\omega) - \varphi(\alpha)(\omega)|^p\right] dP(\omega) \end{aligned}$$

ここで被積分関数を $g(u, \omega)$ とおく。すなわち

$g(u, \omega) = \left[|u^{\frac{1}{p}}\varphi(1)(\omega) + \varphi(\alpha)(\omega)|^p + |u^{\frac{1}{p}}\varphi(1)(\omega) - \varphi(\alpha)(\omega)|^p\right]$

このとき

$$f(u) = \int g(u, \omega) dP(\omega)$$

$\omega \in \Omega$ を固定して考えれば $\varphi(1)(\omega), \varphi(\alpha)(\omega)$ は実数値であるから $\forall \omega \in \Omega$（固定）について

$$\begin{cases} g''(u, \omega) \geq 0, & 1 \leq p \leq 2 \\ g''(u, \omega) \geq 0, & 2 \leq p < \infty \end{cases}$$

従って、$f''(u) = \int g''(u, \omega) dP(\omega)$ についても成り立つ。

Lemma 4 (Kigami, Okazaki and Takahashi[2])

$u_1, u_2, \ldots, u_n \geq 0, w_1, w_2, \ldots, w_n \in \mathbf{C}, |w_i| = 1$

$F(u_1, u_2, \ldots, u_n) = \mathbf{E}\left[|\varepsilon_i w_i u_i^{\frac{1}{p}}|^p\right]$

$\Rightarrow$ $1 \leq p \leq 2$ $F(u)$ は各 u_i について下に凸

$2 \leq p < \infty$ $F(u)$ は各 u_i について上に凸

Theorem（重み付き n 要素 Hanner 不等式）

n を自然数、$\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n$ を独立な Rademacher 列, $w_1, w_2, \ldots, w_n \in \mathbf{C}$ $|w_i| = 1$, $x_1, x_2, \ldots, x_n \in \boldsymbol{L}_{\mathbf{R}}^{\mathrm{p}}$ とする。

$1 \leq p \leq 2 \Rightarrow \mathbf{E}\left\|\sum_{i=1}^{n} \varepsilon_i w_i x_i\right\|^p \geq \mathbf{E}\left|\sum_{i=1}^{n} \varepsilon_i w_i \|x_i\|\right|^p$

$$2 \leq p < \infty \Rightarrow \mathbf{E}\left\| \sum_{i=1}^{n} \varepsilon_i w_i x_i \right\|^p \leq \mathbf{E}\left| \sum_{i=1}^{n} \varepsilon_i w_i \| x_i \| \right|^p$$

Proof.

(i) $1 \leq p \leq 2$ とする。

$$\begin{aligned}
\mathbf{E}\left\| \sum_{i=1}^{n} \varepsilon_i w_i x_i \right\|^p &= \mathbf{E}\left[\int_S \left| \sum_{i=1}^{n} \varepsilon_i(\omega) w_i x_i(t) \right|^p d\mu(t) \right] \\
&= \int_S \mathbf{E}\left[\left| \sum_{i=1}^{n} \varepsilon_i(\omega) w_i x_i(t) \right|^p \right] d\mu(t) \\
&= \int_S \mathbf{E}\left[\left| \sum_{i=1}^{n} \varepsilon_i(\omega) w_i x_i(t) \right|^p \right] d\mu(t) \\
&= \mathbf{E}\left[\left| \sum_{i=1}^{n} \varepsilon_i w_i \| x_i \| \right|^p \right]
\end{aligned}$$

これより、あらかじめ $x_i(t) \geq 0$ としてよい。Lemma 4 の $F(u_1, u_2, \ldots, u_n)$ に対して Jensen の不等式により

$$\int_S F(x_1(t)^p, x_2(t)^p, \ldots, x_n(t)^p) d\mu(t) \geq F\left(\int_S x_1(t)^p d\mu(t), \int_S x_2(t)^p d\mu(t), \ldots, \int_S x_n(t)^p d\mu(t) \right)$$

即ち

$$\begin{aligned}
\textit{left-hand side} &= \int_S \mathbf{E}\left[\left| \sum_{i=1}^{n} \varepsilon_i w_i x_i(t) \right|^p \right] d\mu(t) \\
&= \mathbf{E}\left[\int_S \left| \sum_{i=1}^{n} \varepsilon_i w_i x_i(t) \right|^p d\mu(t) \right] \\
&= \mathbf{E}\left[\left\| \sum_{i=1}^{n} \varepsilon_i w_i x_i \right\|_{L_R^p}^p \right]
\end{aligned}$$

$$\textit{right-hand side} = \mathbf{E}\left[\left| \sum_{i=1}^{n} \varepsilon_i w_i \| x_i \| \right|^p \right]$$

より求める不等式を得ている。

(i) $2 \leq p < \infty$

上に凸な関数について Jensen の不等式により、(i) の場合と逆向きの不等式を得る。

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NORMS OF SOME SINGULAR INTEGRAL OPERATORS ON WEIGHTED L^2 SPACES

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Abstract. Let α and β be bounded functions on the unit circle T, and let W be a positive function on T such that $W, \log W \in L^1$. Then the singular integral operator $S_{\alpha,\beta}$ is defined by $S_{\alpha,\beta}f = \alpha P_+ f + \beta P_- f, (f \in L^2(W))$ where P_+ is an analytic projection and $P_- = I - P_+$. Let h be an outer function such that $W = |h|^2$, and let $\phi = \bar{h}/h$. We give three formulas of the norm of $S_{\alpha,\beta}$ on $L^2(W)$ using α, β and ϕ. If α, β are constant functions, then our results contain the Feldman-Krupnik-Marcus theorem.

1. $F(x)$ の不動点と $S_{\alpha,\beta}$ の第1ノルム公式

単位円周 $T := \{\zeta; |\zeta| = 1\}$ 上の正規化されたLebesgue測度 $dm(\zeta) := d\theta/2\pi$ （$\zeta = e^{i\theta}$）と正値可積分関数 W について $L^2(W)$ のノルムを　$\|f\|_{L^2(W)} := \left\{\int_T |f|^2 W dm\right\}^{1/2}$ と定め可積分関数 f の特異積分 Sf を

$$(Sf)(\zeta) := \frac{1}{\pi i}\int_T \frac{f(\eta)}{\eta - \zeta}d\eta \quad (a.e.\zeta \in T),$$

（積分は Cauchy の主値積分）と定める。$P_+ := (I+S)/2$,　$P_- := (I-S)/2$（I は恒等作用素），$\alpha, \beta \in L^\infty$ について特異積分作用素 $S_{\alpha,\beta}$ を $S_{\alpha,\beta}f := \alpha P_+ f + \beta P_- f$ と定める。

問題　$\|S_{\alpha,\beta}\|_{L^2(W)} := \sup_{\|f\|_{L^2(W)}=1} \|S_{\alpha,\beta}f\|_{L^2(W)}$ を計算するための公式を求めよ。

荷重関数 W が定数のときは，すべての $\alpha, \beta \in L^\infty$ について $\|S_{\alpha,\beta}\|_{L^2} < \infty$ であり，我々はその正確な値を計算するための公式を以前に発表し [15] にまとめた。そのときの主定理が系 1 である。一方，W が定数でなくても $\|P_+\|_W < \infty$ であれば，すべての $\alpha, \beta \in L^\infty$ について $\|S_{\alpha,\beta}\|_{L^2} < \infty$ である。α, β が定数のとき，系 2 のような Feldman-Krupnik-Marcus によるノルム公式が知られており，今回の我々とは別の方法で証明されていた（cf. [3], [7, Section 13.5]）。我々は W, α, β が定数でない場合にも適用できる第 1 ノルム公式（定理 1）を求めた。その証明は Cotlar-Sadosky の lifting 定理と Hilbert 空間の議論による。第 2 ノルム公式（定理 2）は定理 1 を用いて証明でき，系 2 を含んでいる。第 3 ノルム公式（定理 3）は定理 2 のようにそれから系 2 が直ちに導かれるというわけではないが，$\alpha\bar{\beta} \notin H^\infty$ の場合にも適用できるノルム公式を与えており，定理 2 と同じく定理 1 を用いて証明できる。Koosis の定理 [9] より，$\|S_{\alpha,\beta}\|_{L^2(W)} < \infty$ を満たす相異なる関数 α, β が存在するための必要十分条件は $W^{-1} \in L^1$ である (cf. [14])。よって，$W^{-1} \notin L^1$ かつ $\|S_{\alpha,\beta}\|_{L^2(W)} < \infty$ ならば $\alpha \equiv \beta$ となり，$S_{\alpha,\beta}$ は掛け算作用素になり $\|S_{\alpha,\beta}\|_{L^2(W)} = \|\alpha I\|_{L^2(W)} = \|\alpha\|_\infty$ は

よく知られている。よって $W^{-1} \in L^1$ の場合を調べればよい。$W > 0, W \in L^1$ であったから，$\log W \in L^1$ となる。よって，outer 関数 $h \in H^2$ により $W = |h|^2$ と書ける。このとき $\phi := \bar{h}/h$ と定めると，$\|P_+\|_{L^2(W)}$ や $\|S\|_{L^2(W)}$ が有限であるための必要十分条件は $\inf_{k\in H^\infty} \|\phi - k\|_\infty < 1$ であることは，Helson-Szegö の定理としてよく知られている。このとき，すべての $\alpha, \beta \in L^\infty$ について $\|S_{\alpha,\beta}\|_{L^2(W)} < \infty$ となる。しかし，Koosis の定理により $\inf_{k\in H^\infty} \|\phi - k\|_\infty < 1$ でなくても $W^{-1} \in L^1$ であれば $\|S_{\alpha,\beta}\|_{L^2(W)} < \infty$ となるような $\alpha, \beta \in L^\infty$ はたくさんある (cf. [14])。このとき，次の第 1 ノルム公式が成り立つ。

定理 1 $\alpha, \beta \in L^\infty$ とする。outer 関数 $h \in H^2$ により $\phi := \bar{h}/h$，$W := |h|^2$ とおき $F(x)$ を

$$F(x) := \inf_{k\in H^\infty} \left\| \frac{|\alpha|^2 + |\beta|^2}{2} + \sqrt{|x - \alpha\bar{\beta} - \bar{\phi}k|^2 + \left(\frac{|\alpha|^2 - |\beta|^2}{2}\right)^2} \right\|_\infty$$

と定義すると，infimum は attain し，$F(x)$ は実変数 x の convex 関数であることがわかる。このとき，次の (1), (2) が成り立つ。

(1) もし $\|S_{\alpha,\beta}\|_{L^2(W)} < \infty$ ならば，$F\left(\|S_{\alpha,\beta}\|^2_{L^2(W)}\right) = \|S_{\alpha,\beta}\|^2_{L^2(W)}$ が成り立つ。

(2) もし $\inf_{k\in H^\infty} \|\phi - k\|_\infty < 1$ ならば，$\|S_{\alpha,\beta}\|_{L^2(W)} < \infty$ であり，方程式 $F(x) = x$ は唯一の解 $x = \|S_{\alpha,\beta}\|^2_{L^2(W)}$ を持つ。

次に，関数 $y = F(x)$ の具体例をいくつか考える。

例 1. W が定数のときは ϕ も定数になり，

$$F(x) = \inf_{k\in H^\infty} \left\| \frac{|\alpha|^2 + |\beta|^2}{2} + \sqrt{|x - \alpha\bar{\beta} - k|^2 + \left(\frac{|\alpha|^2 - |\beta|^2}{2}\right)^2} \right\|_\infty$$

$$= \inf_{k\in H^\infty} \left\| \frac{|\alpha|^2 + |\beta|^2}{2} + \sqrt{|\alpha\bar{\beta} - k|^2 + \left(\frac{|\alpha|^2 - |\beta|^2}{2}\right)^2} \right\|_\infty = F(0).$$

よって $F(x)$ は定数になり，定理 1 (1) より $F(x) = F\left(\|S_{\alpha,\beta}\|^2_{L^2(W)}\right) = \|S_{\alpha,\beta}\|^2_{L^2(W)}$.
よって次の系 1 が成り立つ。

系 1 $\alpha, \beta \in L^\infty$ かつ W が定数のとき，

$$\|S_{\alpha,\beta}\|^2_{L^2} = \inf_{k\in H^\infty} \left\| \frac{|\alpha|^2 + |\beta|^2}{2} + \sqrt{|\alpha\bar{\beta} - k|^2 + \left(\frac{|\alpha|^2 - |\beta|^2}{2}\right)^2} \right\|_\infty .$$

例 2. W が定数かつ $\alpha\bar{\beta} \in H^\infty$ のときは，例 1 より $F(x)$ は定数になり，$F(x) = F(\|S_{\alpha,\beta}\|^2_{L^2}) = \|S_{\alpha,\beta}\|^2_{L^2}$. 定理 1 や系 1 の infimum は $k = \alpha\bar{\beta}$ で attain し，$\|S_{\alpha,\beta}\|_{L^2} = \max\{\|\alpha\|_\infty, \|\beta\|_\infty\}$.

例 3.　定理 1 の条件の下で $c := \inf_{k\in H^\infty} \|\phi-k\|_\infty$ とおく。$\alpha=\beta=0$ のとき，$F(x)=c|x|$. もし $c=1$ ならば，$F(x)=x$ の解は一意でない。もし $c<1$ ならば，$F(x)=x$ の解は $x=0$. よって，定理 1(1) より，$\|S_{0,0}\|_{L^2(W)}=0$.

例 4.　同じく，$\alpha=1,\beta=0$ のとき，$F(x)=\frac{1}{2}+\sqrt{c^2x^2+\frac{1}{4}}$. もし $c=1$ ならば，$F(x)=x$ は解を持たない。よって，定理 1(1) より，$\|S_{1,0}\|_{L^2(W)}$ も $\|P_+\|_{L^2(W)}$ も有限でない。もし $c<1$ ならば，$F(x)=x$ の解は $x=\frac{1}{1-c^2}$. よって，定理 1(1) より，

$$\|S_{1,0}\|_{L^2(W)}=\|P_+\|_{L^2(W)}=\sqrt{x}=\frac{1}{\sqrt{1-c^2}}.$$

例 5.　同じく，$\alpha=1,\beta=-1$ のとき，$F(x)=1+c|x+1|$. もし $c=1$ ならば，$F(x)=x$ は解を持たない。よって，定理 1(1) より，$\|S_{1,-1}\|_{L^2(W)}$ も $\|S\|_{L^2(W)}$ も有限でない。もし $c<1$ ならば，$F(x)=x$ の解は $x=\frac{1+c}{1-c}$. よって，定理 1(1) より，

$$\|S_{1,-1}\|_{L^2(W)}=\|S\|_{L^2(W)}=\sqrt{x}=\sqrt{\frac{1+c}{1-c}}.$$

例 4 のノルム公式は，古くから知られていた (cf. [8], [13])。Ljance [13] は，$range(P_+)$ と $range(P_-)$ のなす角 θ について

$$\|P_+\|_{L^2(W)}=\|P_-\|_{L^2(W)}=\frac{1}{\sin\theta}$$

を示し，Helson-Szegö [8] は $c=\cos\theta$ を示した。その後，Spitkovsky [18], [19] は

$$\|P_+\|_{L^2(W)}=\frac{1}{2}\left(\|S\|_{L^2(W)}+\frac{1}{\|S\|_{L^2(W)}}\right),$$

$$\|S\|_{L^2(W)}=\|P_+\|_{L^2(W)}+\sqrt{\|P_+\|^2_{L^2(W)}-1}=\cot\frac{\theta}{2}$$

を示し，例 5 のノルム公式を求めた。その後，我々は Cotlar-Sadosky の lifting 定理を用いた別証明を与えた。

2. $S_{\alpha,\beta}$ の第 2，第 3 ノルム公式

$\alpha\in H^\infty, \bar{\beta}\in H^\infty$ のときや，α,β の一方が 0 のときなどは，$\alpha\bar{\beta}\in H^\infty$ が成り立っている。このときは次の第 2 ノルム公式が成り立つ。

定理 2　定理 1 の条件の下で，$\alpha\bar{\beta}\in H^\infty$，$|\alpha(\zeta)-\beta(\zeta)|>0$ のとき，

$$\|S_{\alpha,\beta}\|_{L^2(W)}=\inf_{k\in H^\infty,|\phi-k|<1}\left\|\sqrt{\gamma_k+\left(\frac{|\alpha|+|\beta|}{2}\right)^2}+\sqrt{\gamma_k+\left(\frac{|\alpha|-|\beta|}{2}\right)^2}\right\|_\infty.$$

$$\text{ただし}\quad \gamma_k := \left|\frac{\alpha-\beta}{2}\right|^2\left(\frac{1}{1-|\phi-k|^2}-1\right).$$

このように第 2 ノルム公式は α,β の対称式になっているから，積 $\alpha\bar{\beta}$が定数のときは

$$\|S_{\alpha,\beta}\|_{L^2(W)} = \|S_{\beta,\alpha}\|_{L^2(W)}$$

が成り立つ。この等式は α,β が定数のときに次の系 2 （Feldman-Krupnik Marcus のノルム公式）から知られていた。一方，系 1 より $\alpha(\zeta)=\zeta, \beta(\zeta)=1, W(\zeta)=1$ のとき，この等式は成り立たない。よって，この等式は $\alpha\bar{\beta}\in H^\infty$ のとき一般的には成り立たない。

系 2　α,β が定数，$c=\inf_{k\in H^\infty}\|\phi-k\|_\infty<1$ のとき，

$$\|S_{\alpha,\beta}\|_{L^2(W)} = \sqrt{\gamma+\left(\frac{|\alpha|+|\beta|}{2}\right)^2}+\sqrt{\gamma+\left(\frac{|\alpha|-|\beta|}{2}\right)^2},$$

$$\text{ただし，}\quad \gamma := \left|\frac{\alpha-\beta}{2}\right|^2\left(\frac{1}{1-c^2}-1\right) = \left|\frac{\alpha-\beta}{2}\right|^2\left(\|P_+\|^2_{L^2(W)}-1\right).$$

右辺の各項に $\sqrt{\gamma+a^2}\le\sqrt{\gamma}+|a|$ を使うと次の不等式を得る。

$$\max\{|\alpha|,|\beta|\}\le\|S_{\alpha,\beta}\|_{L^2(W)}\le\max\{|\alpha|,|\beta|\}+|\alpha-\beta|\sqrt{\|P_+\|^2_{L^2(W)}-1}.$$

同様にして，$\alpha\bar{\beta}\in H^\infty$ のとき次を得る。

系 3　定理 2 の条件の下で $c=\inf_{k\in H^\infty}\|\phi-k\|_\infty<1$ のとき，系 2 と同じように γ を定めると，γ は関数になり，

$$\max\{\|\alpha\|_\infty,\|\beta\|_\infty\}\le\|S_{\alpha,\beta}\|_{L^2(W)}\le\left\|\sqrt{\gamma+\left(\frac{|\alpha|+|\beta|}{2}\right)^2}+\sqrt{\gamma+\left(\frac{|\alpha|-|\beta|}{2}\right)^2}\right\|_\infty$$

$$\le\max\{\|\alpha\|_\infty,\|\beta\|_\infty\}+\|\alpha-\beta\|_\infty\sqrt{\|P_+\|^2_{L^2(W)}-1}.$$

例 6.　$\alpha(\zeta)=\zeta+1, \beta(\zeta)=1, W(\zeta)=|\zeta+1|^{1/2}$ のとき，$\alpha\bar{\beta}\in H^\infty, \phi(e^{i\theta})=e^{-i\theta/4}, \zeta=e^{i\theta}, -\pi\le\theta<\pi$. よって，$|\phi-\frac{1}{\sqrt{2}}|\le\frac{1}{\sqrt{2}}<1$. よって，定理 2 より

$$2\le\|S_{\alpha,\beta}\|_{L^2(W)}=\inf_{k\in H^\infty,|\phi-k|<1}\left\|\sqrt{\gamma_k+\left(\frac{|\alpha|+|\beta|}{2}\right)^2}+\sqrt{\gamma_k+\left(\frac{|\alpha|-|\beta|}{2}\right)^2}\right\|_\infty$$

$$\le\left\|\sqrt{\gamma_{1/\sqrt{2}}+\left(\frac{|\alpha|+1}{2}\right)^2}+\sqrt{\gamma_{1/\sqrt{2}}+\left(\frac{|\alpha|-1}{2}\right)^2}\right\|_\infty.$$

$$\text{ただし}\quad \gamma_{1/\sqrt{2}}=\frac{1}{4}\left(\frac{1}{1-|\phi-\frac{1}{\sqrt{2}}|^2}-1\right).\quad \text{よって，}$$

$$2 \leq \|S_{\alpha,\beta}\|_{L^2(W)} \leq \sup_{-\pi \leq \theta < \pi} \left\{ \sqrt{\frac{3-2\sqrt{2}\cos(\theta/4)}{4(2\sqrt{2}\cos(\theta/4)-1)} + \left(\frac{\sqrt{2(1+\cos\theta)}+1}{2}\right)^2} \right.$$
$$\left. + \sqrt{\frac{3-2\sqrt{2}\cos(\theta/4)}{4(2\sqrt{2}\cos(\theta/4)-1)} + \left(\frac{\sqrt{2(1+\cos\theta)}-1}{2}\right)^2} \right\}$$
$$= \sqrt{\frac{29+2\sqrt{2}}{14}} + \sqrt{\frac{1+2\sqrt{2}}{14}} = 2.03...$$

このとき，$\|P_+\|_{L^2(W)} = \sqrt{2}$ (cf. [12]) より，$\gamma = \left|\frac{\alpha-\beta}{2}\right|^2 (\|P_+\|^2_{L^2(W)} - 1) = \frac{1}{4}$. よって，

$$\left\| \sqrt{\gamma + \left(\frac{|\alpha|+|\beta|}{2}\right)^2} + \sqrt{\gamma + \left(\frac{|\alpha|-|\beta|}{2}\right)^2} \right\|_\infty$$
$$= \sqrt{\frac{1}{4} + \left(\frac{3}{2}\right)^2} + \sqrt{\frac{1}{4} + \left(\frac{1}{2}\right)^2} = \frac{\sqrt{10}+\sqrt{2}}{2} = 2.28...$$

よって，

$$2 \leq \|S_{\alpha,\beta}\|_{L^2(W)} < \left\| \sqrt{\gamma + \left(\frac{|\alpha|+|\beta|}{2}\right)^2} + \sqrt{\gamma + \left(\frac{|\alpha|-|\beta|}{2}\right)^2} \right\|_\infty .$$

これは，系 3 の等号が成り立たない例を与えている。

系 4　定理 1 の条件の下で $|\alpha(\zeta)| > 0$ のとき，

$$\|\alpha P_+\|_{L^2(W)} = \inf_{k \in H^\infty} \left\| \frac{\alpha}{\sqrt{1-|\phi-k|^2}} \right\|_\infty .$$

系 5　定理 1 の条件の下で，$\varepsilon_n \in L^\infty, \varepsilon_n > 0, \|\varepsilon_n\|_\infty \to 0, (n \to \infty)$ を満たす $\{\varepsilon_n\}$ が存在して

$$\|\alpha P_+\|_{L^2(W)} = \lim_{n\to\infty} \|(|\alpha| + \varepsilon_n) P_+\|_{L^2(W)} = \lim_{n\to\infty} \inf_{k\in H^\infty} \left\| \frac{|\alpha|+\varepsilon_n}{\sqrt{1-|\phi-k|^2}} \right\|_\infty .$$

もし $\inf_{k\in H^\infty} \|\phi - k\|_\infty < 1$ ならば $\varepsilon_n = \frac{1}{n}$ ととれる。

例 7.　$W(\zeta) = |\zeta+1|^{1/2}, h(\zeta) = (\zeta+1)^{1/2}, \zeta = e^{i\theta}, \phi(\zeta) = \overline{h(\zeta)}/h(\zeta) = e^{-i\theta/4}$, $E \subset T$ のとき，系 4 と系 5 より

$$\|\chi_E P_+\|_{L^2(W)} = \lim_{n\to\infty} \|(\chi_E + \frac{1}{n}) P_+\|_{L^2(W)} = \lim_{n\to\infty} \inf_{k\in H^\infty} \left\| \frac{\chi_E + \frac{1}{n}}{\sqrt{1-|\phi-k|^2}} \right\|_\infty$$

$$\leq \left\| \frac{\chi_E + 0.1}{\sqrt{1 - |e^{-i\theta/4} - \frac{1}{\sqrt{2}}|^2}} \right\|_\infty = \left\| \frac{\chi_E + 0.1}{\sqrt{\sqrt{2}\cos(\frac{\theta}{4}) - \frac{1}{2}}} \right\|_\infty .$$

$-\pi \leq \theta < \pi$ のとき $\frac{1}{\sqrt{2}} \leq \cos(\frac{\theta}{4}) \leq 1$. よって

$$\sqrt{\frac{2}{2\sqrt{2}-1}} \leq \frac{1}{\sqrt{\sqrt{2}\cos(\frac{\theta}{4}) - \frac{1}{2}}} \leq \sqrt{2}.$$

よって，E^c 上では

$$\frac{\chi_E + 0.1}{\sqrt{\sqrt{2}\cos(\frac{\theta}{4}) - \frac{1}{2}}} \leq 0.1 \cdot \sqrt{2} < 0.15.$$

また $\theta = 0$ のとき,

$$\frac{1}{\sqrt{\sqrt{2}\cos(\frac{\theta}{4}) - \frac{1}{2}}} = \sqrt{\frac{2}{2\sqrt{2}-1}} = 1.04... < 1.05.$$

よって，十分小さい $\varepsilon > 0$ について $E = (-\varepsilon, \varepsilon)$ と定めると,

$$\|\chi_E P_+\|_{L^2(W)} < 1.05 \cdot 1.1 = 1.155 < \sqrt{2} = \|\chi_E\|_\infty \|P_+\|_{L^2(W)}$$

よって，$\|\alpha P_+\|_{L^2(W)} = \|\alpha\|_\infty \|P_+\|_{L^2(W)}$ は一般的には成り立たない。

次に，$\|S_{\alpha,\beta}\|_{L^2(W)}$ と $\|S\|_{L^2(W)}$ を比べることを考える。$\alpha, \beta \in L^\infty$ について，$S_{\alpha,\beta} = \frac{\alpha+\beta}{2} I + \frac{\alpha-\beta}{2} S$ より,

$$\|S_{\alpha,\beta}\|_{L^2(W)} \leq \left\| \frac{\alpha+\beta}{2} \right\|_\infty + \left\| \frac{\alpha-\beta}{2} \right\|_\infty \|S\|_{L^2(W)}$$

はよく知られている。特に $\alpha\bar{\beta} \in H^\infty$ のときは定理 2 より次が成り立つ。

系 6 定理 2 の条件の下で

$$\max\{\|\alpha\|_\infty, \|\beta\|_\infty\} \leq \|S_{\alpha,\beta}\|_{L^2(W)} \leq \max\{\|\alpha\|_\infty, \|\beta\|_\infty\} \|S\|_{L^2(W)}.$$

例 8. $\alpha(\zeta) = 1, \beta(\zeta) = \zeta$ のとき $\alpha\bar{\beta} \notin H^\infty$. 系 1 より,

$$\|S_{\alpha,\beta}\|_{L^2}^2 = \inf_{k \in H^\infty} \left\| \frac{|\alpha|^2 + |\beta|^2}{2} + \sqrt{|\alpha\bar{\beta} - k|^2 + \left(\frac{|\alpha|^2 - |\beta|^2}{2}\right)^2} \right\|_\infty = 1 + \inf_{k \in H^\infty} \|\alpha\bar{\beta} - k\|_\infty = 2.$$

よって $\|S_{\alpha,\beta}\|_{L^2} = \sqrt{2}$. 一方，$\|S\|_{L^2} = 1$ より $\|S_{\alpha,\beta}\|_{L^2} > \|S_{\alpha,\beta}\|_{L^2}$. よって，系 6 は $\alpha\bar{\beta} \notin H^\infty$ のとき一般的には成り立たない。更に，定理 2 で $\alpha\bar{\beta} \in H^\infty$ をなくせると仮定すると，系 6 でもそれをなくせることが導かれ矛盾。よって，定理 2 も $\alpha\bar{\beta} \notin H^\infty$ のとき一般的には成り立たない。次の第 3 ノルム公式はそのときも成り立つ。

定理 3 定理 1 の条件の下で，$||\alpha(\zeta)| - |\beta(\zeta)|| > 0$ のとき,

$$\|S_{\alpha,\beta}\|_{L^2(W)}^2 = \inf_{k \in H^\infty, |\phi - k| < 1} \left\| \frac{|\alpha|^2 + |\beta|^2 - 2Re(\bar{\alpha}\beta(1 - \bar{\phi}k))}{1 - |\phi - k|^2} \right\|_\infty .$$

$\boxed{\text{系 7}}$ 定理 3 の条件の下で，

$$(1)\qquad \inf_{k\in H^\infty}\left\|\frac{|\alpha|-|\beta|}{\sqrt{1-|\phi-k|^2}}\right\|_\infty \le \inf_{k\in H^\infty}\left\|\frac{\max\{|\bar\beta-\bar\alpha(1-\bar\phi k)|,|\alpha-\beta(1-\bar\phi k)|\}}{\sqrt{1-|\phi-k|^2}}\right\|_\infty$$

$$\le \|S_{\alpha,\beta}\|_{L^2(W)} \le \inf_{k\in H^\infty}\left\|\frac{|\alpha|+|\beta|}{\sqrt{1-|\phi-k|^2}}\right\|_\infty .$$

更に $S_{\alpha,\beta}=(\alpha-\beta)P_+ + \beta I = S_{\alpha-\beta,0}+\beta I,\quad S_{\alpha,\beta}=(\beta-\alpha)P_- + \alpha I = S_{0,\beta-\alpha}+\alpha I$ より，

$$\max\{\|\alpha\|_\infty,\|\beta\|_\infty\} \le \|S_{\alpha,\beta}\|_{L^2(W)} \le \|\alpha-\beta\|_\infty\|P_+\|_{L^2(W)} + \min\{\|\alpha\|_\infty,\|\beta\|_\infty\}$$

が知られているが，この精密化として

$$(2)\qquad \max\left\{\|\alpha\|_\infty,\|\beta\|_\infty,\inf_{k\in H^\infty}\left\|\frac{|\alpha|-|\beta|}{\sqrt{1-|\phi-k|^2}}\right\|_\infty\right\}$$

$$\le \|S_{\alpha,\beta}\|_{L^2(W)} \le \inf_{k\in H^\infty}\left\|\frac{\alpha-\beta}{\sqrt{1-|\phi-k|^2}}\right\|_\infty + \min\{\|\alpha\|_\infty,\|\beta\|_\infty\}.$$

特に $\beta=0$ のとき，(1) や (2) の不等式は次のように系 4 のノルム公式になる。

$$(3)\qquad \|\alpha P_+\|_{L^2(W)} = \inf_{k\in H^\infty}\left\|\frac{\alpha}{\sqrt{1-|\phi-k|^2}}\right\|_\infty .$$

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Positive cone の中での infimum の問題

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Problem of Infima in the Positive Cone

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Abstract We consider two positive (semi-definite) operators A, B on a Hilbert space. We prove that the infimum $A \wedge B$ in the cone of positive operators exists if and only if $[A]B \geq [B]A$ or $[A]B \leq [B]A$. In this case $A \wedge B$ in $\mathcal{P}$ is $\min\{[A]B, [B]A\}$. Here $[A]B$ is the A-absolutely continuous part of B.

1　**背景と問題**　Hilbert 空間 $\mathcal{H}$ の有界な selfadjoint な作用素の全体は普通の順序、すなわち

$$A \geq B \quad \Longleftrightarrow \quad A - B \quad positive\ (semidefinite)$$

に関して ordered vector space になるが、束 (lattice) 構造からはほど遠い。実際 A, B に infimum $A \wedge B$ が存在するのは $A \geq B$ か $A \leq B$ 、すなわち A, B が comparable な場合に限られる。

この事情を納得するために、$A, B \geq 0$ が可換の場合を考えよう。同時対角化で

$$A = \int_a^b f(\lambda) dE(\lambda), \quad B = \int_a^b g(\lambda) dE(\lambda)$$

と表示できる。ここで $\{E(\lambda); a \leq \lambda \leq b\}$ は projection の増加系で、$f(\lambda), g(\lambda)$ は non-negative な可測関数である。

もし infimum $A \wedge B$ が存在すれば、それは

$$C \stackrel{def}{=} \int_a^b \min\{f(\lambda), g(\lambda)\} dE(\lambda)$$

となることは想像できるであろう。いま $f(\lambda) > 0, g(\lambda) > 0$ であるが、$f \geq g$ でも $f \leq g$ でもなければ、$D \geq 0$ で $D \leq A, B$ で、$C \not\geq D$ なものが構成できる。これは 2×2 行列

$$\begin{bmatrix} \alpha & 0 \\ 0 & \beta \end{bmatrix}, \quad \begin{bmatrix} \alpha' & 0 \\ 0 & \beta' \end{bmatrix}$$

で、$\alpha > \alpha' > 0, \beta' > \beta > 0$ のとき、2×2 行列 $X = [x_{ij}], x_{12} \neq 0$ で

$$0 \leq \begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{bmatrix} \leq \begin{bmatrix} \alpha & 0 \\ 0 & \beta \end{bmatrix}, \begin{bmatrix} \alpha' & 0 \\ 0 & \beta' \end{bmatrix}$$

ではあるが

$$\begin{bmatrix} \alpha' & 0 \\ 0 & \beta \end{bmatrix} \not\geq \begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{bmatrix}$$

のものが構成できることと本質的に同じである。

考察の対象を positive (semi-definite) な作用素のなす錐 (cone) $\mathcal{P}$ に制限すると、少し状況が変わる。

上で説明したように、$A, B > 0$ 、すなわち positive invertible で、comparable でないときは、$\mathcal{P}$ での infimum $A \wedge B$ は存在しない。

invertible でない場合で、$A \geq 0$ が rank 1 なら $0 \leq X \leq A$ な X はすべて A の (positive) scalar 倍であるから、どの $B \geq 0$ に対しても $\{X\ ;\ 0 \leq X \leq A, B\}$ は全順序集合となり最大元 、すなわち $\mathcal{P}$ での infimum $A \wedge B$ が存在する。この逆もまた成り立つ。すなわち、どの $B \geq 0$ に対しても $\mathcal{P}$ での infimum $A \wedge B$ が存在するのは、A が rank ≤ 1 のときに限る。

さらによく知られたように、P, Q が projection であれば、$\mathcal{P}$ での infimum $P \wedge Q$ が存在し、それは projection で

$$\mathrm{ran}(P \wedge Q) = \mathrm{ran}(P) \cap \mathrm{ran}(Q)$$

で決められる。この事実は次のように一般化できる。すなわち, Krein の公式によればどの $B \geq 0$ に対しても最大元

$$\max\{X; 0 \leq X \leq B, \mathrm{ran}(X) \subset \mathrm{ran}(P)\}$$

が存在する。いま $0 \leq B \leq I$ であれば、上で $0 \leq X \leq I$ となるので、$\mathrm{ran}(X) \subset \mathrm{ran}(P)$ は $X \leq P$ となる。このことは上で決まる最大元が $\mathcal{P}$ での infimum $P \wedge B$ になる。すなわち、 projection P と $0 \leq B \leq I$ にたいして、$\mathcal{P}$ での infimum $P \wedge B$ が存在する。

Moreland - Gudder [2] は、Hilbert 空間が有限次元の場合、$0 \leq A, B \leq I$ に $\mathcal{P}$ での infimum $A \wedge B$ が存在するための必要十分条件を確立した。

この講演の目的は、より transparent な方法で、一般の Hilbert 空間の場合にこの問題を解決することにある。

2　**可換な場合への還元**　$0 \le A, B \le I$ を考えるが、$ker(A+B) = \{0\}$ と仮定しても一般性を失わない。このとき $\{X; 0 \le X \le A+B\}$ から $\{Y; 0 \le Y \le I\}$ への bijection φ が

$$X = (A+B)^{1/2}\varphi(X)(A+B)^{1/2}$$

で決まる。φ は affine, order bijection になっている。したがって $\mathcal{P}$ で infimum $A \wedge B$ が存在することと、$\mathcal{P}$ での infimum $\varphi(A) \wedge \varphi(B)$ が存在することは同値で、

$$\varphi(A \wedge B) = \varphi(A) \wedge \varphi(B)$$

となる。重要なのは

$$\varphi(A) + \varphi(B) = I$$

であるから、$\varphi(A), \varphi(B)$ が **可換** となることである。$\varphi(A), \varphi(B)$ の同時対角化を行うと

$$\varphi(A) = \int_0^1 \lambda dE(\lambda), \quad \varphi(B) = \int_0^1 (1-\lambda) dE(\lambda)$$

と書かれる。infimum の候補は

$$\tilde{C} \stackrel{def}{=} \int_0^1 \min\{\lambda, 1-\lambda\} dE(\lambda)$$

であるが、これは $\{X; 0 \le X \le \varphi(A), \varphi(B)\}$ の中での極大元であることは容易に判る。もし $\mathcal{P}$ での infimum $\varphi(A) \wedge \varphi(B)$ が存在すれば、$\tilde{C}$ と一致しなくてはならない。ここで

$$\tilde{C} = \int_{0+}^{1-} \max\{\lambda, 1-\lambda\} dE(\lambda)$$

と書かれることに注目しよう。これは

$$0 \le X \le \varphi(A), \varphi(B) \iff 0 \le X \le \int_{0+}^{1-} \lambda dE(\lambda), \int_{0+}^{1-} (1-\lambda) dE(\lambda)$$

と合わせると、

$$\int_{0+}^{1-} \lambda dE(\lambda) \quad \text{と} \quad \int_{0+}^{1-} (1-\lambda) dE(\lambda)$$

の $\mathcal{P}$ での infimum の存在を問題にすることと同じであることがわかる。

$(0,1)$ で $\lambda > 0, 1-\lambda > 0$ であるから、上に説明したと同様に、$\mathcal{P}$ での問題の infimum が存在するのは

$$\int_{0+}^{1-} \lambda dE(\lambda) \quad \text{と} \quad \int_{0+}^{1-} (1-\lambda) dE(\lambda)$$

が comparable であるときに限られることがわかる。このときは当然

$$\varphi(A) \wedge \varphi(B) \ (in \ \mathcal{P}) = \min\{\int_{0+}^{1-} \lambda dE(\lambda), \int_{0+}^{1-} (1-\lambda) dE(\lambda)\}$$

となる。したがって問題は、

$$\int_{0+}^{1-} \lambda dE(\lambda) \quad \text{や} \quad \int_{0+}^{1-} (1-\lambda) dE(\lambda)$$

を $\varphi(A), \varphi(B)$ から直接どのように構成できるかにある。ここで鍵となるのは

$$\begin{aligned} \int_{0+}^{1-} \lambda dE(\lambda) &= \sup_n \int_0^1 \frac{n(1-\lambda)\lambda}{\lambda + n(1-\lambda)} dE(\lambda), \\ \int_{0+}^{1-} (1-\lambda) dE(\lambda) &= \sup_n \int_0^1 \frac{n(1-\lambda)\lambda}{n\lambda + (1-\lambda)} dE(\lambda) \end{aligned}$$

の表示である。この表示の意味を考えるために一般論に戻ろう。

3　一般の場合への復元　$X, Y \geq 0$ に対して一般には infimum はないが、infimum もどきものがある。それは X, Y の **並列和** (parallel sum) と言われ $X : Y$ と書かれるもので、可逆な場合は

$$X : Y = \{X^{-1} + Y^{-1}\}^{-1}$$

で定義されるが、一般の場合は二次形式として

$$\langle (X : Y)a, a \rangle = \inf\{\langle Xb, b\rangle + \langle Yc, c\rangle ; a = b + c\}$$

で定義される。$X : Y \leq X, Y$ は明らかである。これを一歩進めて Y の X-**絶対連続部分** (X-absolutely continuous part) $[X]Y$ を

$$[X]Y = \lim_{n\to\infty} (nX) : Y$$

で定義する (Ando [1]) 。容易に判ることとして、$Y \longmapsto [X]Y$ は単調増加であり

$$0 \leq Y \leq X \implies [X]Y = Y$$

がでる。Projection P に関しては $[P]I = P$ となる。

さらに

$$[X]Y = [[X]Y]Y = [X:Y]Y$$

が知られている。

まえの場合にもどると、明らかに

$$[\varphi(B)]\varphi(A) = \int_{0+}^{1-} \lambda dE(\lambda), \quad [\varphi(A)]\varphi(B) = \int_{0+}^{1-} (1-\lambda) dE(\lambda)$$

となり、また

$$\varphi([B]A) = [\varphi(B)]\varphi(A) \quad \varphi([A]B) = [\varphi(A)]\varphi(B)$$

が確かめられので、前節の結果は次の形にまとめられる。

定理 $A, B \geq 0$ のとき、$\mathcal{P}$ での infimum $A \wedge B$ が存在する必要十分条件は

$$[A]B \geq [B]A \quad \text{または} \quad [A]B \leq [B]A$$

が成り立つことである。このとき明らかに

$$\begin{aligned} A \wedge B \ (in\ \mathcal{P}) &= \min\{[A]B, [B]A\} \\ &= \min\{[A:B]A, [A:B]B\}. \end{aligned}$$

4 既知の場合の再考

projection P と $0 \leq B \leq I$ に対しては

$$\begin{aligned} P \wedge B &= [P]B \\ &\leq [[P]B]([P]B) \\ &= [[P]B]([P]I) \\ &= [[P]B]P = [B]P \end{aligned}$$

であるから

$$P \wedge B \ (in\ \mathcal{P}) = [P]B$$

となる。特に projection P, Q にたいしては

$$P \wedge Q\ (in\ \mathcal{P}) = [P]Q = [Q]P$$

となる。

$A \geq 0$ が rank 1 のときは、$B \geq 0$ にたいして

$$0 \leq [A]B \leq (scalar)A, \quad [B]A \leq A$$

となるから、$[A]B$ と $[B]A$ は comparable である。

5　有限次元の場合　有限次元の場合は $\mathrm{ran}(A), \mathrm{ran}(B)$ はすべて閉部分空間となり、これらへの projection を P_A, P_B であらわすと

$$\alpha A \leq P_A \leq \beta A, \quad \alpha B \leq P_B \leq \beta B$$

な $\alpha, \beta > 0$ があるから

$$[A]B = [P_A]B, \quad [B]A = [P_B]A$$

と projection を使って書かれる。さらに $\mathrm{ran}(A) \cap \mathrm{ran}(B)$ への projection を $P_{A,B}$ と書くと、有限次元性を使って $\mathrm{ran}(A : B) = \mathrm{ran} \cap \mathrm{ran}(B)$ がでるので、上のことから

$$[A : B]A = [P_{A,B}]A = A \wedge P_{A,B}(in \mathcal{P}), \quad [A : B]B = [P_{A,B}]B = B \wedge P_{A,B}(in \mathcal{P})$$

となるから、$\mathcal{P}$ での infimum $A \wedge B$ が存在するとめの必要十分条件は、$A \wedge P_{A,B}$ と $B \wedge P_{A,B}$ が comparable のことである。そしてその小さい方が $\mathcal{P}$ での infimum $A \wedge B$ となる。これが Moreland and Gudder の結果である。

6　参考文献

[1] T. Ando, Lebesgue type decompositon of positive operators , Acta Sci. Math. (Szeged) 38 (1976), 61–67.

[2] T. Moreland and S. Gudder, Infima of Hilbert space effects, (preprint).

Survey of unitary ρ-dilation and ρ-radius

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1．はじめに

ここではユニタリー ρ-拡大に関する結果の紹介を目的とするが、新しい結果は中路氏（北海道大学・理学部）との共同研究による。以下では A, B はヒルベルト空間 $\mathcal{H}$ 上の有界線形作用素とする。

$\rho > 0$ に対して A が ρ-縮小作用素であるとは $\mathcal{K} \supset \mathcal{H}$ なるヒルベルト空間 $\mathcal{K}$ とその上のユニタリ作用素 U があって、

$$A^n = \rho P U^n|_{\mathcal{H}} \quad (n = 1, 2, \cdots)$$

が成り立つこととする。ここで P は $\mathcal{K}$ から $\mathcal{H}$ への直交射影作用素である ([14] 参照)。ρ-縮小作用素の特徴づけとして次のことが知られている。

Theorem A. *(B.Sz.-Nagy and C. Foiaş[15])* $A \in B(\mathcal{H})$, $\rho > 0$ とする。このとき、次の条件は同値である。

(i) A がρ-縮小作用素

(ii) $r(A) \leq \frac{\rho}{|\rho - 1|}$, $\|zA\{\rho - z(\rho-1)A\}^{-1}\| \leq 1$ $(|z| < 1)$

(iii) $-2Re[zA(I - zA)^{-1}] \leq \rho I$ $(|z| < 1)$

$\|A\|$ を A の作用素ノルム：$\|A\| = \sup\{\|Ax\| \mid \|x\| = 1\}$ とすると、$\|A\| \leq 1$ (A を縮小作用素と呼ぶ) であるとき、1953 年に Sz.-Nagy によって A は 1-縮小作用素であることが示された (Sz.-Nagy[12] 参照)。このことからも、J.von Neumann[16] が示した、A が縮小作用素のとき、すべての多項式　$f(z)$ に対して、

$$\|f(T)\| \leq \max_{|z| \leq 1} |f(z)|$$

が成り立つことが示される。また、1965 年に Berger[4] が 数域半径：$w(A) = \sup\{|(Ax, x)| \mid \|x\| = 1\}$ が 1 である必要十分条件は A が 2-縮小作用素となることであることを示した。このことを一般化して、Sz.-Nagy and Foiaş が、1966 年に、先に述べたρ-縮小作用素を定義した。

2．$B(\mathcal{H})$ の部分集合としてのρ-縮小作用素

$\mathcal{C}_\rho := \{A \in B(\mathcal{H}) \mid A : \rho - 縮小作用素\}$

$S := \{A \in B(\mathcal{H}) \mid A : 縮小作用素に相似\}$、すなわち、

$$\exists C\ (\|C\| \leq 1), \exists S(invertible)\ ;\ A = S^{-1}CS$$

$\mathcal{PLB} = \{A \in B(\mathcal{H}) \mid A : \text{polynomially bounded}\}$ 、すなわち、

$$\exists M > 0 \; ; \; ||p(A)|| \leq M \sup_{|z|\leq 1} |p(z)| \text{ for all polynomial } p$$

$\mathcal{PB} := \{A \in B(\mathcal{H}) \mid A : \text{power bounded}\}$ 、すなわち、

$$\exists M > 0 \; ; \; \sup_{n\geq 1} ||A^n|| \leq M$$

$\mathcal{R} = \{A \in B(\mathcal{H}) \mid r(A) \leq 1\}$ ($r(A)$ は A のスペクトル半径)
とする。

このとき、次の関係がいえる。

$$\cup_{\rho>0}\mathcal{C}_\rho \subsetneqq \mathcal{S} \subsetneqq \mathcal{PLB} \subsetneqq \mathcal{PB} \subsetneqq \mathcal{R}$$

これらの関係について歴史的なことを少し説明をしておく。1966 年に B.Sz.-Nagy and C.Foiaş は $\cup_{\rho>0}\mathcal{C}_\rho \subset \mathcal{S}$ であることを示した。それに先だって、1959 年に Sz.-Nagy[13] が $\mathcal{PB} \subset \mathcal{S}$ が成り立つか、という問題を出した。Sz.-Nagy は A^{-1} が存在し、$A, A^{-1} \in \mathcal{PB}$ ならば、$A \in \mathcal{S}$ であることを示した（実際にはユニタリー作用素に相似を示した。）。また、Sz.-Nagy and Foiaş によって $A \in \mathcal{PB}$ で作用素ならば、$A \in \mathcal{S}$ が知られている（もっと強く、$A \in \mathcal{R}$ でコンパクトならば $A \in \mathcal{S}$ であることが示されていて、このことから最初の包含関係は真であることがわかる。）。これに対して 1964 年に S.Foguel[5] はこの問題に関して反例を与えた。1968 年に A.Lebow[8] は Foguel の例は polynomially bounded でないことを示し、1970 年に P.R.Halmos[7] は $\mathcal{PB} \subset \mathcal{S}$ かという問題（Halmos Problem）を提起した。この問題に関しては、1996 年に G.Pisier[17] が成り立たないことを示した。
なお、$\cup_{\rho>0}\mathcal{C}_\rho$ は $\mathcal{R}$ で norm dense になることも容易に示せる。

$\mathcal{C}_\rho$ は $z \in \mathbb{C}$, $|z| = 1$ に対して、$z\mathcal{C}_\rho \subset \mathcal{C}_\rho$ がいえるので、Minkowski functional が考えられ、A の ρ-半径を $w_\rho(A) = \inf\{\gamma > 0 \mid \gamma^{-1}A \in \mathcal{C}_\rho\}$ で定義する (J.A.R.Holbrook[7] 参照）。このとき、$w_\rho(\cdot)$ は $0 < \rho \leq 2$ でノルムになるが $2 < \rho < \infty$ のときは以下のように準ノルムにはなるがノルムではない。

$$w_\rho(A+B) \leq \frac{\rho}{2}\{w_\rho(A) + w_\rho(B)\}$$

ρ-半径は次の性質をもつ。(T.Ando[1] 参照)

(1) $$w_1(A) = \|A\| : \text{ the operatoer norm}$$

$$(2) \qquad w_2(A) = w(A): \text{ the numerical radius}$$

$$(3) \qquad \lim_{\rho \longrightarrow \infty} w_\rho(A) = r(A): \text{ the spectral radius}$$

$$(4) \qquad \begin{aligned} &\log w_{\lambda\rho+(1-\lambda)\sigma}(A) \\ &\qquad \leq \lambda \log w_\rho(A) + (1-\lambda) \log w_\sigma(A) \end{aligned}$$

$$(5) \qquad 1 \leq \sigma \leq \rho \text{ ならば } w_\rho(A) \leq w_\sigma(A)$$

$$(6) \qquad 1 \leq \sigma \leq \rho \text{ ならば } \sigma w_\sigma(A) \leq \rho w_\rho(A)$$

$$(7) \qquad w_\rho(UAU^*) = w_\rho(A) \quad (\text{unitary } U)$$

$w|_\rho(\cdot)$ は Schwarz norm である。すなわち analytic function $f : \overline{\mathbb{D}} \to \overline{\mathbb{D}}(\mathbb{D} := \{z \in \mathbb{C} \mid |z| < 1\}$, $f(0) = 0$ に対して

$$(8) \qquad w_\rho(A) \leq 1 \implies w_\rho(f(A)) \leq 1$$

$S \geq 0$ を有界半正値作用素とするとき、C.-K.Li, N.-K.Tsing and F.Uhlig[9] により 一般化された数域 $V_S(A)$ が次のように定義された。

$$(9) \qquad V_S(A) = \{(Ax, x) \mid x \in \mathcal{H},\ |(Sx, x)| = 1\}$$

$V_S(A)$ に関して $v_S(A)$ を次で定義する。

$$(10) \qquad v_S(A) = \sup\{|(Ax, x)| \mid x \in \mathcal{H},\ |(Sx, x)| = 1\}$$

特に $S = I$ のときは、$V_I(A) = W(A) := \{(Ax, x) \mid ||x|| = 1\}$($A$ の数域) で、かつ $v_I(A) = w(A)$ である。
この応用としては、A がコンパクト作用素で $S = |A| := (A^*A)^{1/2}$ 、$v_S(A) \leq 1$ のとき、A が正規作用素であることが T.Ando and K.Takahashi[3] によって知られている。$0 \leq \lambda, \mu \leq 1$ に対して

$$\mathbf{w}_\lambda(A) = \sup\{|(Ax, x)| \mid \lambda||x||^2 + (1-\lambda)||Ax||^2 \leq 1\}$$

$$\mathbf{w}_\mu^+ = \sup\{\mu||Ax||^2 + (1-\mu)|(Ax, x)| \mid ||x|| \leq 1\}$$

として、$\mu \geq 1$ に対して、

$$\mathbf{w}_\mu^-(A) = \sup\{\mu||Ax||^2 + (\mu-1)|(Ax,x)| \mid ||x|| \leq 1\}$$

とする。ここでは、A が ρ-縮小作用素であることを A に関係する $S \geq 0$ と $v_S(A)$ を使って、さらに、$\mathbf{w}_\lambda(A)$, $\mathbf{w}_\mu^+(A)$, $\mathbf{w}_\mu^-(A)$ を用いて特徴づける。
また、$w_\rho(A)$ を $|(Ax,x)|$, $||Ax||$, $||x||$ を使って表示する。

3. $w_\rho(A)$ の表示

$\rho > 0$ に対して A がρ-縮小作用素であるための条件として次を得る。

定理 1　$\rho > 0$, $\rho \neq 1$ とする。$0 < t \leq 1$ に対して

$$S_t = \frac{1}{t}\frac{\rho}{2|\rho-1|}I + t\frac{\rho-2}{2|\rho-1|}|A|^2 \tag{11}$$

とする。このとき、A がρ-縮小作用素であるための必要十分条件は、$S_t \geq 0$ かつ、$v_{S_t}(A) \leq 1$ が $0 < t \leq 1$ で成り立つことである。
証明は$\rho > 0$ とするとき、$A \in C_\rho$ であるための必要十分条件が

$$||x||^2 + \left(1-\frac{2}{\rho}\right)|\zeta|^2||Ax||^2 - 2\left(1-\frac{1}{\rho}\right)Re\zeta(Ax,x) \geq 0$$

($\zeta \in \mathbb{D}$, $x \in \mathcal{H}$) であることから、容易に導かれる。
この結果として、

系 2　$\rho > 0$, $\rho \neq 1$ に対して A が ρ-縮小作用素であるために必要十分条件は、$0 < t \leq 1$ に対して $A = S_t^{1/2}B_tS_t^{1/2}$ とできることである。ここで、B_t は $w(B_t) \leq 1$ である。(S_t は (11) で定義された作用素)

特に、$0 < \rho \leq 2$, $\rho \neq 1$ のときは、次のことが言える。

定理 3 $0 < \rho \leq 2$, $\rho \neq 1$ とすると、A が ρ-縮小作用素であるために必要十分条件は $A = S^{1/2}BS^{1/2}$ となることである。ただし、$S = (\rho I + (\rho-2)|A|^2)/2|\rho-1|$ で、B は $w(B) \leq 1$ を満たす。

系 4　$\rho > 0$, $\rho \neq 1$ とする。$t \geq w_\rho(A)$ であるとき、

$$|(Ax,x)| \leq t\frac{\rho}{2|\rho-1|}||x||^2 + \frac{1}{t}\frac{\rho-2}{2|\rho-1|}||Ax||^2 \ (x \in \mathcal{H}) \tag{12}$$

が成り立つ。逆に、ある t_0 があって、$t \geq t_0$ に対して (12) が成り立てば、$t_0 \geq w_\rho(A)$ である。
この事実は後で用いる。

系 5
(1) $0 \leq \mu \leq 1$, $1 \leq \rho = 2/(\mu+1) \leq 2$ とする。このとき、A が ρ-縮小作用素であることと $\mathrm{w}_\mu^+(A) \leq 1$ であることが同値である。
(2) $1 \leq \mu$, $0 < \rho = 2/(\mu+1) \leq 1$ とする。このとき、A が ρ-縮小作用素であることと $\mathrm{w}_\mu^-(A) \leq 1$ であることが同値である。

$\mu||A||^2 + \lambda w(A) \leq 1$ ($\mu + \lambda = 1$ $(0 \leq \mu \leq 1)$ または$\mu - \lambda = 1$ $(1 \leq \mu < \infty)$) とする。仮に $\rho = 2/(\mu+1)$ ならば $w_\rho(A) \leq 1$ が系5から言えるが、逆は言えない。

系 6 $0 < \rho \leq 2$ とする。このとき $w_\rho(A) \leq 1$ であるための必要十分条件は $w(\mu|A|^2 + \lambda^{i\theta}A) \leq 1$ $(0 < \theta \leq 2\pi)$。ここで $\mu + \lambda = 1$, $\mu = \frac{2}{\rho} - 1$ または $\mu - \lambda = 1$, $\mu = \frac{2}{\rho} - 1$ とする。

定理 7 $0 \leq \lambda \leq 1$ とする。
(1) $\frac{1}{2} \leq \lambda \leq 1$, $\rho = 2\lambda/(2\lambda - 1) \geq 2$ とするとき、A がρ-縮小作用素である必要十分条件は $\mathrm{w}_\lambda(tA) \leq 1$ $(0 < t \leq 1)$ となることである。
(2) $0 \leq \lambda \leq \frac{1}{2}$, $1 \leq \rho = 2(\lambda-1)/(2\lambda-1) \geq 2$ かつ A が可逆とするとき、A^{-1} がρ-縮小作用素である必要十分条件は $\mathrm{w}_\lambda(tA) \leq 1$ $(t \geq 1)$ となることである。
この証明は定理1を用いてできる。

定理 8 $0 < \rho$, $\rho \neq 1$ とする。このとき、

$$w_\rho(A) = \frac{|\rho - 1|}{\rho} \sup\{|(Ax, x)| + \sqrt{D} \mid ||x|| = 1,\ D \geq 0\} \tag{13}$$

ただし、$D = |(Ax,x)|^2 - \frac{\rho(\rho-2)}{(\rho-1)^2}||Ax||^2$ とする。
証明 系4より、$t \geq w_\rho(A)$ とすると$\lambda||x||^2t^2 - |(Ax,x)|t + (1-\lambda)||Ax||^2 \geq 0$ $(x \in \mathcal{H})$ となる。ここで$\lambda = \rho/2|\rho - 1|$。$D \geq 0$ と仮定すると、

$$w_\rho(A) \leq \frac{|(Ax,x)| - \sqrt{D}}{2\lambda||x||^2}$$

または、

$$w_\rho(A) \geq \frac{|(Ax,x)| + \sqrt{D}}{2\lambda||x||^2}$$

である。

$$t_0 = \sup_{x \neq 0} \frac{|(Ax,x)| + \sqrt{D}}{2\lambda||x||^2}$$

とおくと、$t \geq t_0$ ならば

$$\lambda||x||^2t^2 - |(Ax,x)|t + (1-\lambda)||Ax||^2 \geq 0$$

が $x \in \mathcal{H}$ でいえる。系４から、$w_\rho(A) \leq t_0$ となる。
$0 < \rho \leq 2$ のとき、$|(Ax,x)| - \sqrt{D} \leq 0$ だから、$w_\rho(A) \geq t_0$ がいえて、$w_\rho(A) = t_0$ となる。
$\rho > 2$ のとき、$x_0 \in \mathcal{H}$ で

$$w_\rho(A) \leq \frac{|(Ax_0,x_0)| - \sqrt{D}}{2\lambda||x_0||^2}$$

をみたすとすると、系４から

$$|(Ax_0,x_0)| \leq t\lambda||x_0||^2 + \frac{1}{t}(1-\lambda)||Ax_0||^2$$

$(t > \{|(Ax_0,x_0)| - \sqrt{D}\}/2\lambda||x_0||^2)$ がいえる。一方で、もし、$t > \{|(Ax_0,x_0)| - \sqrt{D}\}/2\lambda||x_0||^2$ ならば、

$$\lambda||x_0||^2t^2 - |(Ax_0,x_0)|t + (1-\lambda)||Ax_0||^2 < 0$$

となり、これは矛盾する。したがって、

$$w_\rho(A) \geq \frac{|(Ax,x)| + \sqrt{D}}{2\lambda||x||^2}$$

がすべての $x \in \mathcal{H}$ について成り立つ。ゆえに、$\rho > 2$ でも $w_\rho(A) = t_0$ がいえる。

系 ９ $0 < \rho \leq 2$ とすると、次の不等式が成り立つ。

$$\max\left\{2\left|1-\frac{1}{\rho}\right|w(A),\ \sqrt{\frac{2-\rho}{\rho}}||A||\right\} \leq w_\rho(A) \leq 2\left|1-\frac{1}{\rho}\right|w(A) + \sqrt{\frac{2-\rho}{\rho}}||A||$$

先に注意したように、$0 < \rho \leq 2$ では$\rho w_\rho(A) = (2-\rho)w_{2-\rho}(A)$ が成り立つから、$1 \leq \rho \leq 2$ でこの不等式がいえることを示すとよいことがわかる。定理８から簡単にこの不等式は示される。また、$1 < \rho \leq 2$ で、左辺の不等式はすでに知られている不等式

$$w(A) \leq w_\rho(A),\ ||A|| \leq \rho w_\rho(A)$$

より、良くないが、左辺との関連で挙げておいた。

定理 １０
(1) $0 < \rho \leq 2$ とする。このとき、

$$w_\rho(A) = \frac{2}{\rho}\sup_{||x||=1}\sup_{0\leq t\leq 1}\{\sqrt{\rho(2-\rho)}||Ax||\sqrt{t(1-t)} + |\rho-1||(Ax,x)|t\}$$

(2) $\rho > 2$ のとき、

$$w_\rho(A) = \frac{2}{\rho} \sup_{||x||=1, D\geq 0} \inf_{t\geq 1} \{-\sqrt{\rho(2-\rho)}||Ax||\sqrt{t(t-1)} + |\rho-1||(Ax,x)|t\}$$

ただし、$D = |(Ax,x)|^2 - \frac{\rho(\rho-2)}{(\rho-1)^2}||Ax||^2$ とする。

証明 (1) $0 < \rho \leq 2$ とする。$||x|| = 1$ なる $x \in \mathcal{H}$ に対して $g(t,x) = \sqrt{\rho(2-\rho)}||Ax||\sqrt{t(1-t)} + |\rho-1| \cdot |(Ax,x)|t$ とおく。このとき、

$$\begin{aligned} &2\left(\frac{d}{dt}g(t,x)\right)\sqrt{t(1-t)} \\ &= \sqrt{\rho(2-\rho)}||Ax||(1-2t) + 2|\rho-1| \cdot |(Ax,x)|\sqrt{t(1-t)} \end{aligned}$$

となり、$\frac{d}{dt}g(t,x)_{t=t_0} = 0,\ 0 \leq t_0 \leq 1$ である必要十分条件は

$$t_0 = \frac{1}{2} + \frac{|(Ax,x)|}{2\sqrt{D}}$$

であり、したがって、

$$\sqrt{t_0(1-t_0)} = \frac{\sqrt{\rho(2-\rho)}||Ax||}{2|\rho-1|\sqrt{D}}$$

となる。ここで $D = |(Ax,x)|^2 - \frac{\rho(\rho-2)}{(\rho-1)^2}||Ax||^2$ とする。ゆえに

$$\begin{aligned} &\frac{2}{\rho} \sup_{||x||=1} \sup_{0\leq t\leq 1} \{\sqrt{\rho(2-\rho)}||Ay||\sqrt{t(1-t)} + |\rho-1| \cdot |(Ax,x)|t\} \\ &= \frac{2}{\rho} \sup_{||x||=1} \left\{\sqrt{\rho(2-\rho)}||Ax|| \times \frac{\sqrt{\rho(2-\rho)}||Ax||}{2|\rho-1| \cdot \sqrt{D}} + |\rho-1| \cdot |(Ax,x)| \times \left(\frac{1}{2} + \frac{|(Ax,x)|}{2\sqrt{D}}\right)\right\} \\ &= \frac{|\rho-1|}{\rho} \sup_{||x||=1} \{|(Ax,x)| + \sqrt{D}\} \end{aligned}$$

(2) も $f(t,x) = -\sqrt{\rho(\rho-2)}||Ax||\sqrt{t(t-1)} + (\rho-1)|(Ax,x)|t$ とおいて、(1) と同様な考察で示される。

定理 10 のことから、R.Mathias and K.Okubo[10] の次の表示が得られる。

系 1 1 $0 < \rho \leq 2$ とする。このとき、$w_\rho(A) = \frac{2}{\rho}w(C_\rho \otimes A)$ となる。ここで、$C_\rho = \begin{bmatrix} 0 & \sqrt{\rho(2-\rho)} \\ 0 & 1-\rho \end{bmatrix}$ である。

4．2×2 の場合のρ-縮小作用素の条件と応用

ρ-半径を計算することはそんなに簡単ではない。ここでは、$\mathcal{H}$ の次元が 2 の場合について考える。

定理 1 2 $a,b \in \overline{\mathbb{D}} := \{z \in \mathbb{C} \mid |z| \leq 1\}$ とする。このとき、$\rho \geq 1$ とするとき、$A = \begin{bmatrix} a & c \\ 0 & b \end{bmatrix}$ が ρ-縮小作用素であるための必要十分条件は

$$\begin{aligned} &|c|^2 + |a-b|^2 \\ &\leq \inf_{\zeta \in \mathbb{D}} \left| \frac{\{\rho + (1-\rho)\overline{a}\overline{\zeta}\}\{\rho + (1-\rho)b\zeta\} - \overline{a}b|\zeta|^2}{\rho\zeta} \right|^2 \end{aligned}$$

である。

証明 $A = \begin{bmatrix} a & c \\ 0 & b \end{bmatrix}$ とする。$w_\rho(A) \leq 1$ であるための必要十分条件は $||g_\zeta(A)|| \leq 1$ $(\zeta \in \mathbb{D})$ となることである。ここで、$g_\zeta(z) = z\zeta/(\rho + (1-\rho)z\zeta)$ である ([2] 参照)。$a = b$ のときは、G.Misra の結果 [11] より、$||g_\zeta(A)|| \leq 1$ である必要十分条件が

$$|c||g'_\zeta(a)| \leq \left\{ \sup\{|f'(g'_\zeta(a))| \mid f \in Hol(\overline{\mathbb{D}}, g_\zeta(a), \overline{\mathbb{D}})\} \right\}^{-1}$$

ここで、$Hol(\overline{\mathbb{D}}, g_\zeta(a), \overline{\mathbb{D}}) = \{f : \overline{\mathbb{D}} \longrightarrow \overline{\mathbb{D}} \mid f : holomorphic,\ f(g_\zeta(a)) = 0\}$ とする。このことから、$|c| \leq \frac{|\rho + (1-\rho)a\zeta|^2 - |a\zeta|^2}{|\zeta\rho|}$ がでる。一方、$a \neq b$ のとき、$g_\zeta(a) \neq g_\zeta(b)$ $(\zeta \neq 0)$ であり、$w_\rho(A) \leq 1$ であるための必要十分条件は、

$$\left\| \begin{bmatrix} f \circ g_\zeta(a) & \frac{f \circ g_\zeta(a) - f \circ g_\zeta(b)}{a-b} \\ 0 & f \circ g_\zeta(b) \end{bmatrix} \right\| \leq 1$$

が任意の $f \in Hol(\overline{\mathbb{D}}, g_\zeta(a), \overline{\mathbb{D}})$, $\zeta \in \overline{\mathbb{D}}$ で成り立つことである。これは、再び Misra の結果を使うと

$$\frac{|c|^2}{|a-b|^2} \leq \frac{1}{|f \circ g_\zeta(a)|^2}$$

$(f \in Hol(\overline{\mathbb{D}}, g_\zeta(a), \overline{\mathbb{D}}),\ \zeta \in \overline{\mathbb{D}})$ と同値になり、これを計算すると定理が得られる。

$A \in B(\mathcal{H})$ が quadratic operator であるとは、A が quadratic polynomial, すなわち、ある $r, s \in \mathbb{C}$ に対して $A^2 + rA + sI = 0$ をみたすこととする。$\mathcal{H}$ が有限次元のとき、仮に、$t^2 + rt + s = (t-a)(t-b)$ とすると、A は

$$\begin{bmatrix} aI_l & C \\ 0 & bI_m \end{bmatrix}, \text{ここで } C \in M_{lm} \tag{14}$$

にユニタリー相似である。ρ-縮小作用素はユニタリー相似によって不変であるから、quadratic operator は (14) の形であると考えてよい。このとき、定理 12 の一般化として次のことがいえる。

定理 1 3 $A = \begin{bmatrix} aI_l & C \\ 0 & bI_m \end{bmatrix}$ とすると、A が ρ-contraction であるための必要十分条件は

$$||C||^2 + |a-b|^2 \le \inf_{\zeta \in D} \left| \frac{\{\rho + (1-\rho)\overline{a\zeta}\}\{\rho + (1-\rho)b\zeta\} - \overline{a}b|\zeta|^2}{\rho\zeta} \right|^2$$

となることである。

以下では $A = \begin{bmatrix} a & c \\ 0 & b \end{bmatrix}$ とする。このとき、定理12結果を用いて計算すると、次のことがいえる。ただし、$\rho \ge 1$ とする。

(1) $a = b$ のとき

$$w_\rho(A) = \frac{2(\rho-1)|a| + |c| + \sqrt{(2|a|-|c|)^2 + 4\rho|a||c|}}{2\rho}$$

(2) $a > 0,\ b \ge 0$ のとき

$$w_\rho(A) = \Bigg\{(\rho-1)(a+b) + \sqrt{|c|^2 + (a-b)^2} + \sqrt{\left((1-\rho)((a+b) - \sqrt{|c|^2+(a-b)^2}\right)^2 - 4\rho(\rho-2)ab}\Bigg\}/2\rho$$

(3) $a = -b,\ a > 0$ のとき

$$w_\rho(A) = \frac{\sqrt{4\rho(2-\rho)a^2 + |c|^2 + a^2} + \sqrt{|c|^2 + 4a^2}}{2\rho} = \frac{w(A) + \sqrt{w(A)^2 + \rho(2-\rho)a^2}}{\rho}$$

(3) のことは、T.Ando and K.Nishio [2] でも指摘されている。

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A Note on Kolmogorov's counterexample.

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$(\Omega, \mathcal{A}, P)$ は確率空間とする.

$M_n \equiv \{\phi_n = 1\}$, $M_n' \equiv \{\phi_n = -1\}$, $e_n \equiv \{\phi_n \neq \pm 1\}$ $(n = 1, 2, 3, \cdots\cdots)$

$\{\phi_n\}$ は独立、同分布、の実確率変数列

とし、

$$\Omega = M_n + M_n' + e_n$$

$$P(M_n) = P(M_n') = \frac{1}{2} \quad (\therefore P(e_n) = 0,\ P(M_n) \neq 0,\ P(M_n') \neq 0)$$

とする. この $\{\phi_n\}$ はいわゆる Rademacher の関数列を少し一般にしたものである.

$$\Omega = M_{n+k} + M_{n+k}' + e_{n+k} \quad (k \geqq 1)$$

$$\therefore \underbrace{\Omega \setminus e_n = M_{n+k} + M_{n+k}' + (e_{n+k} \setminus e_n) (= M_n + M_n')}.$$

ここで $0 = P(e_{n+k}) \geqq P(e_{n+k} \setminus e_k) \geqq 0$ より $P(e_{n+k} \setminus e_n) = 0$.

又、$\phi = M_{n+k} \cap e_{n+k} \supset M_{n+k} \cap (e_{n+k} \setminus e_n)$ より

$M_{n+k} \cap (e_{n+k} \setminus e_n) = \phi$. 同様に $M_{n+k}' \cap (e_{n+k} \setminus e_n) = \phi$.

今 $P(\Omega \setminus e_n) = P(\Omega) - P(e_n) = 1 - 0 = 1$ であるから、$n = n_0$ を1つ取り $e_n (= e_{n_0})$ を fix して、以後 Ω の代りに $\Omega \setminus e_n$ (確率空間 Ω の制限) の上で考えることにする.

$\mathcal{A}_n \equiv \sigma(\phi_1, \phi_2, \ldots, \phi_n)$. $(n \geqq n_0)$.

ここで $\mathcal{A}_n \supseteq \sigma(M_n, M_n') (= \{\phi, M_n, M_n', \Omega \setminus e_{n_0}\})$

となるから $\underbrace{\mathcal{A}_n \neq \{\phi, \Omega \setminus e_{n_0}\}}$. (或いは n を大きく取ってもよい.)

$$\phi_{n+1} \underset{a.e.}{=\!=} E(\phi_{n+1} / \mathcal{A}_{n+1}) = c_1' \cdot \chi_{M_{n+1}} + c_2' \cdot \chi_{M_{n+1}'} + c_3' \cdot \chi_{e_{n+1} \setminus e_n}$$

$$\underset{a.e.}{=\!=} c_1' \cdot \chi_{M_{n+1}} + c_2' \cdot \chi_{M_{n+1}'}.$$

$$\therefore c_1' = 1, \quad c_2' = -1.$$

従って. 仮に $\phi_{n+1} \perp \mathcal{A}_n$ とすると

$$E(\phi_{n+1} / \mathcal{A}_n) \underset{a.e.}{=\!=} E(\phi_{n+1}) = \underset{\sim}{0}.$$

$\parallel$ a.e.

$$E(\chi_{M_{n+1}} / \mathcal{A}_n) - E(\chi_{M_{n+1}'} / \mathcal{A}_n)$$

$\parallel$

$$E(\chi_{M_{n+1}} / \mathcal{A}_n) - E(\chi_{\Omega \setminus e_n} - \chi_{M_{n+1}} - \chi_{e_{n+1} \setminus e_{n_0}} / \mathcal{A}_n)$$

$$(\because \Omega \setminus e_n = M_{n+1} + M_{n+1}' + e_{n+1} \setminus e_{n_0})$$

$\parallel$

$$E(\chi_{M_{n+1}} / \mathcal{A}_n) - E(\chi_{\Omega} - \chi_{e_n} - \chi_{M_{n+1}} - \chi_{e_{n+1} \setminus e_{n_0}} / \mathcal{A}_n)$$

$$(\because \chi_{\Omega} = \chi_{\Omega \setminus e_n} + \chi_{e_n})$$

$\parallel$ a.e.

$$E(\chi_{M_{n+1}} / \mathcal{A}_n) - \{1 - E(\chi_{M_{n+1}} / \mathcal{A}_n) - \underbrace{E(\chi_{e_n} + \chi_{e_{n+1} \setminus e_{n_0}} / \mathcal{A}_n)}_{\overset{\parallel \, a.e.}{0}}\}$$

$$\left(\begin{array}{l} \because \text{一般に } P(A) = 0 \text{ なら } E(\chi_A / B) = 0 \text{ a.e.} \\ \because) \displaystyle\int_{\Omega} \underbrace{E(\chi_A / B)}_{\overset{\vee\!|}{0}} dP = \int_{\Omega} \chi_A \, dP = P(A) = 0. \end{array}\right)$$

$\parallel$

$$\underbrace{2 \cdot E(\chi_{M_{n+1}} / \mathcal{A}_n) - 1}.$$

$\therefore\ \underbrace{E(\chi_{M_{n+1}}/a_n)}\ \underset{a.e.}{=}\ \frac{1}{2}$

$\qquad \|$

$c_1'' \cdot \chi_{M_n} + c_2'' \cdot \chi_{M_n'} + c_3'' \cdot \chi_{e_{n_0}}$

$\qquad \|$

$\frac{1}{2}\chi_{M_n} + \frac{1}{2}\chi_{M_n'} + c_3'' \cdot \chi_{e_{n_0}}$ $(\because P(M_n) \neq 0,\ P(M_n') \neq 0)$.

$\therefore\ E(\chi_{M_{n+1}}/a_n) = \frac{1}{2}\chi_{M_n} + \frac{1}{2}\chi_{M_n'} = \frac{1}{2}$ on $\Omega \setminus e_{n_0}$.

$\therefore\ a_n \subseteq \{\phi, \Omega \setminus a_{n_0}\}$ 従って $a_n = \{\phi, \Omega \setminus e_{n_0}\}$ となり矛盾.

$\therefore$「ϕ_{n+1} と a_n は独立でない.」

以上の議論は $e_{n_0} = \phi$ の時にも成り立つことに注意する.

$\therefore\ f_n = \sum_{k=1}^{n} \frac{1}{k} \cdot \phi_k$ (Kolmogorov の反例) とすると,

$$E(f_{n+1} - f_n/a_n) = E\left(\frac{1}{n+1} \cdot \phi_{n+1}/a_n\right) = \frac{1}{n+1} \cdot E(\phi_{n+1}/a_n) \neq 0 \text{ on } \Omega \setminus e_{n_0} (\subset \Omega).$$

◎ 従って, (f_n, a_n) は矛盾を含んでいて, マルチンゲールにならない.

(一般の Ω の上で考えても, 除外集合を除いた処で, (f_n) はマルチンゲールにならない.)

注意 一般の Ω の上で f がマルチンゲールになると仮定した場合, σ-代数 a_n は有限生成でないから

$$0 \neq \phi_{n+1} \underset{a.e.}{=} E(\phi_{n+1}/a_n)$$

より ϕ_{n+1} と a_n は独立でない. 従って, f はマルチンゲールにならないという矛盾が出る.

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The Bad Part Of An Outer Function

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Abstract. It is known that an outer function in the Hardy space H^1 can be factored into a product in which one factor is strongly outer and the other is an outer function with the same argument to some inner function. The latter factor is called the bad part of an outer function. We show that the bad part has the following form : $(s+q\bar{s})^2$ where $s \in H^2 \ominus qzH^2$. If $s = 1$ then $s + q\bar{s} = 1 + q$ and if $q = q_1q_2$ then $s + q\bar{s} = q_1 + q_2$ where q_j is an inner function. If s is an outer function then $s + q\bar{s}$ is also outer. We make clear the relation between the order of zeros of the Blaschke part of $s + q\bar{s}$ and q. This is a survey article on the author's two papers [1] and [2].

§1. 問題

Hardy 空間 H^1 の零でない関数 f について、その絶対値 $|f|$ と偏角 $f/|f|$ を知る事は当然重要である。ここで次の二つの問題を考えることは自然である。f と g は H^1 の零でない関数とする。

問題 1 $|f| = |g|$ のとき f と g にはどの様な関係にあるだろうか。

問題 2 $f/|f| = g/|g|$（すなわち $g/f \geq 0$ ）のとき f と g にはどの様な関係にあるだろうか。

問題 1 の解答として、$f = q_1h$ 、$g = q_2h$ かつ $q_1, q_2, h \in H^1$ とできる。ここで q_1, g_2 は inner 関数かつ h は outer 関数である。更に q_1, q_2 は非常に良く研究されている。問題 2 の解答として、$f = t_1h, g = t_2h$ かつ $t_1, t_2, h \in H^1$ とできる。ここで t_1 と t_2 はある inner 関数 q と同じ偏角をもち かつ h は strongly outer 関数である。しかし t_1, t_2 はまだあまり研究されていない。この講演の目的は、t_1 と t_2 を研究することである。

strongly outer 関数ならば outer 関数であるが、逆は成立しない。よって outer 関数についても上の因数分解ができる。その時現われる t_1, t_2 に相当する部分をこの講演ではその表題の bad part と呼ぶ。q が inner 関数のとき q と $(1+q)^2$ の偏角が同じであるので、$(1+q)^2$ は bad part であるが、これは井上氏の結果によって bad part の全てではない。 Helson 氏は $q = q_1q_2$ が inner 関数のとき、q と $(q_1+q_2)^2$ の偏角が同じで、bad part の全てであることを示した。しかし上の因数分解の立場からはこれでもまだ今のところ十分ではない。この講演では、bad part を描き、その自然さを示す。

§2. 定義

D は $\mathbf{C}$ の open unit disc かつ $d\theta/2\pi$ は ∂D 上の normalized Lebesgue measure を示す。$1 \le p \le \infty$ のとき、$L^p = L^p(d\theta/2\pi)$ は Lebesgue space かつ

$$H^p = \{f \in L^p\ ;\ \hat{f}(n) = 0 \quad (n < 0)\}$$

は Hardy space と呼ばれる。H^p は L^p の closed subspace となり、$H^\infty \subset H^2 \subset H^1$ となっている。任意の $f \in H^p$ は D へ analytic extension を持ち かつ f はその boundary value としてとらえることができる。

q が inner function であるとは、$q \in H^1$ かつ $|q| = 1$ *a.e.* on ∂D のことである。$f \in H^1$ かつ $f \not\equiv 0$ とせよ。f が outer function とは、$g \in H^1$、$g/f = q$ かつ $|q| = 1$ *a.e.* on ∂D ならば $q \in H^1$ となることである。このとき q は inner function となる。f が strongly outer function とは、$g \in H^1$、$g/f = \alpha$ かつ $\alpha \ge 0$ *a.e.* on ∂D ならば $\alpha \in H^1$ となることである。このとき (H^1 は定数でない実数値関数を含まないので)、α は定数である。f が stronly outer ならば outer であるが、逆は成立しないことを示すのはやさしい。

§3. 二つの重要な因数分解定理

H^1 は Banach space であり、そのノルムは L^1 のそれであるが、unit ball には extreme point や exposed point が存在する。deLeeuw-Rudin は、次の事を示した。f が outer である必要十分条件は $f/\|f\|_1$ が H^1 の unit ball の extreme point であることである。f が strongly outer である必要十分条件は $f/\|f\|_1$ が H^1 の unit ball の exposed point であることである。この様に outer function や strongly outer function は関数論、作用素論や予測理論などの重要なポイントで顔をだす。この §では outer function と strongly outer function に対する因数分解定理を示す。定理 A は Beurling の有名な定理であり、定理 B は Hayashi [1] の結果である。

定理 A $f \in H^1$ かつ $f \not\equiv 0$ とする。もし f が outer でなければ、$f = qh$ と書ける。ここで q は nonconstant inner であり、h は outer である。

定理 B $f \in H^1$ かつ $f \not\equiv 0$ とする。もし f が outer であるが strongly outer でなければ、$f = th$ と書ける。ここで $h \in H^1$ は strongly outer であり、$t \in H^1$ はある inner function q について、$\bar{q}t \ge 0$ *a.e.* on ∂D である。

定理 A を用いると §1 の問題 1 に対する解答が得られる。定理 B を用いると §1 の問題 2 に対する解答が得られるが、その解答はある意味で十分ではない。

§4. Outer function の bad part

H^1 の一般的な関数の bad part は inner function と見ることができるが（定理 A）、outer function の bad part とは inner function と同じ偏角の関数のことである（定理 B）。outer function の bad part として次の三つのタイプが考えられる。ここで、q, q_1, q_2 は inner かつ $s \in H^2 \ominus qzH^2$ は outer である。(1) $(1+q)^2$、(2) $(q_1+q_2)^2$ かつ (3) $(s+q\bar{s})^2$。 $s=1$ とすると $(s+q\bar{s})^2 = (1+q)^2$ であり、$q = q_1q_2$ かつ $s = q_1$ とすると $(s+q\bar{s})^2 = (q_1+q_2)^2$ である。よって (3) は (1) と (2) を含む。$s \perp qzH^2$ だから $q\bar{s} \in H^2$、よって $(s+q\bar{s})^2 \in H^1$ である。また $\bar{q}(s+q\bar{s})^2 = (\bar{q}s+\bar{s})(s+q\bar{s}) = |s+q\bar{s}|^2 \geq 0$. §1 に触れた様に、(1) と (2) では bad part の全体を描いていないが、(3) はその全体を描いている事を我々は主張したい。

$\boxed{\text{定理 1}}$ [3]。q を inner function とせよ。$t \in H^1$ が outer function かつ $\bar{q}t \geq 0$ である必要十分条件は $t = (s+q\bar{s})^2$ と書けることである。ここで $s \in H^2 \ominus qzH^2$ かつ s は outer function である。

証明. 十分性。もし $s \in H^2 \ominus qzH^2$ とすると、上の注意により、$t = (s+q\bar{s})^2 \in H^1$ かつ $\bar{q}t \geq 0$ である。s が outer とすると、ある inner q_0 に対して $q\bar{s} = q_0 s$ と書ける。よって $t = (1+q_0)^2 s^2$ となり、t は outer となる。必要性。$t \in H^1$ が outer かつ $\bar{q}t \geq 0$ とすると、ある outer $t_0 \in H^2$ が存在して $t = t_0^2$ と書ける かつ $\bar{q}t_0 = \overline{t_0}$ 。$s = t_0/2$ とおくと $t = (s+q\bar{s})^2$ かつ s は outer である。

§5. H^1 の極値問題

$\phi \in L^\infty$ に対して、

$$K_\phi(f) = \int_0^{2\pi} \phi(e^{i\theta}) f(e^{i\theta}) d\theta / 2\pi \quad (f \in H^1)$$

とすると、K_ϕ は H^1 上の continuous linear functional となる。$S_\phi = \{f \in H^1 \ ; \ K_\phi(f) = \|K_\phi\|, \|f\|_1 \leq 1\}$ とする。S_ϕ を描くことは重要である。q_1, q_2 を inner function とするとき、$q_1 \prec q_2$ とは ある $f \in H^1$ に対して $\overline{q_1}q_2 = f/|f|$ と書けるときをいう。

$\boxed{\text{定理 2}}$ [3]。$S_\phi \neq \emptyset$ ならば inner function q と strongly outer function g_0 が存在して、

$$S_\phi = \{f = \gamma q_0 \left(\frac{s+q\bar{s}}{1+q_0}\right)^2 g_0 \text{ かつ } \|f\|_1 = 1\}$$

である。ここで、$\gamma > 0$ は定数、q_0 は $q_0 \prec q$ となる inner function であり かつ $s \in H^2 \ominus qzH^2$ である。

$f \in H^2$ ならば $\forall g \in \bar{z}\bar{H}^2$ に対して

$$\|f+\bar{g}\|_2^2 = \|f\|_2^2 + \|\bar{g}\|_2^2 \geq \|f\|_2^2$$

である。しかし $f \in H^1$ ならば $\forall g \in \bar{z}\bar{H}^1$ に対して、$\|f+\bar{g}\|_1 < \|f\|_1$ となることが起こる。f が inner ならば、$\|f+\bar{g}\|_1 \geq \|f\|_1$ $(\forall g \in \bar{z}\bar{H}^1)$ を見ることはやさしい。更に $f=(s+q\bar{s})^2$ のときも、$\|f+\bar{g}\|_1 \geq \|f\|_1$ $(\forall \in \bar{z}\bar{H}^1)$ が成立する。実際次の事が成立する。

$\boxed{\text{定理 3}}$ [3]。$f \in H^1$ かつ $f \not\equiv 0$ とする。このとき $\|f+\bar{g}\|_1 \geq \|f\|_1 (\forall g \in \bar{z}\bar{H}^1)$ となる必要十分条件は

$$f = q_0 \left(\frac{s+q\bar{s}}{1+q_0}\right)^2$$

となる。ここで q_0 と q は inner function であり、$q_0 \prec q$ かつ $s \in H^2 \ominus qzH^2$ である。

§6. 零点の収束の order

$f \in H^1$ かつ $f \not\equiv 0$ に対して $Z(f;D)=\{z \in D\ ;\ f(z)=0\}$ とする。sing $f = \partial D \cap$ closure of $Z(f;D)$ とする。我々は $Z(f\ ;\ D)$ の点列の sing f への収束の程度 Ord $[f]$ を定義できる。q が infinite Blaschke product かつ sing $q \neq \partial D$ のとき、もし Ord $[q]=\sigma \geq 1$ ならば、この講演で重要であった次の (1) ~ (4) の関数は、Ord $[f] \leq \sigma$ を示すことができる。

(1) $f=(s+q\bar{s})^2 \in H^1$

(2) $f = q_0\left(\dfrac{s+q\bar{s}}{1+q_0}\right)^2 \in H^1, q_0 \prec q$

(3) $f \in H^2 \ominus qzH^2$

(4) f は inner かつ $f \prec q$ 。

この § の結果と正確な定義は preprint [2] にあるが、sing g が finite set である場合は 井上氏による第 5 回関数空間セミナーの報告集に書かれてあります。

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Orders and Norm Topology

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Let E be a partially ordered normed linear space i.e. E has a proper convex cone P which generates E and there exists a norm in E.

We shall consider in this note the following problems.

(1) When the norm $\| \,.\, \|$ is equivalent to an ordered norm.

(2) What is the condition describing sup set for $a, b \in E$ in terms of boundary of convex sets induced by an order P.

A norm $\| \,.\, \|$ is called an ordered norm if $\| x \| \geq \| y \|$ for $x \geq y \geq 0$. In general, a norm on partially ordered linear space E is not nessary an ordered norm. There are many examples of non ordered norm even if E is 2-dimensional .

Let P be an order in E and A be a subset of E. We shall use the notation $P(A) = \{A + P\} \cap \{A - P\}$. It is easy to see that $A \subset P(A)$

Theorem 1 Let E be a partially ordered normed linear space. The norm is equivalent to an ordered norm if and only if $P(U)$ is norm bounded where U is the unit ball of the normed linear space E.

Proof. Let $V = P(U)$. We shall show that if $V \ni x_1, x_2 \geq 0$ and $x_1 \geq x_2$, then $\| x_1 \|_V \geq \| x_2 \|_V$, where $\| \,.\, \|_V$ is a Minkowski functional defined by V.

We shall show that $\alpha x_1 \in V, \alpha \geq 1$ implies $\alpha x_2 \in V$. Since $x_1 = x_2 + p$ for some $p \in P$, $\alpha x_2 = \alpha x_1 - \alpha p \in V - P$. On the other hand $\alpha x_2 = x_2 + (\alpha - 1)x_2 \in V + P$. This means that $x_2 \in V$. Hence $\| x_1 \|_V \geq \| x_2 \|_V$, i.e. the norm $\| \,.\, \|_V$ defined by V is an ordered norm.

Since $P(U) = V$ contains always U, we have the assertion. q.e.d.

Let E be a partially ordered Hausdorff topological linear space with an order P. We assume that P is closed.

We shall define sup set for two elements $a, b \in E$. $a \vee b$ is a set of all minimal elements of $(a+P) \cap (b+P) = U_{a,b}$. Usually $a \vee b$ is not a set of single element.

Definition 1. A subset F of convex cone P is called a (exposed) face of P if there exists a supporting hyperplane of P with $F = P \cap H$.

Definition 2. dim F = dimension of affine hull of F.

Then we have the following theorem.

Theorem 2 If dim F is smaller than 1 for all face of P,
then we have :

$$a \vee b = \partial(a+P) \cap \partial(b+P)$$

where ∂ means relative boundary.

Proof of this theorem is omitted here. We will show proof of Theorem 2 in another paper.

To illustrate Theorem 2, we shall show some exmple.

Example

Let E be all Hermitian operators on 2-dimensional Euclidean space. The order P in E is defined as positive definite order. Then E is considered as 3-dimensional Euclidean space whose element is denoted by (a, b, c) and P is all elements (a, b, c) with $a, b \geq 0$ and $ab \geq c^2$.

It is easy to see that dimension of any face on P is smaller than 1 and so we can see easily how about the set $a \vee b$ by Theorem 2 .

For any $p = (a, b, c)$ with a, b, c being real, we have

$$p \vee 0 = \{(x, y, z); x \geq 0, y \geq 0, x-a \geq 0, y-b \geq 0, xy = z^2, (x-a)(y-b) = (z-c)^2\}.$$

Concerning relations between Riesz space and general partially ordered linear space, we have many interesting results. But, these results will be published in another paper.

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